PS2 solutions

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Problem 1

(a): QEC conditions are that PaP, $Pa^\dagger P$, and $Pa^\dagger a P$ are all proportional to P, where P is the projection on the codespace. First, observe that

$$Pa^\dagger P = (PaP)^\dagger \,, \tag{1}$$

so if the latter is satisfied, so is the former. Observe that the error words

$$a|\bar{0}\rangle = \frac{1}{\sqrt{2}}\left(a|0\rangle + a|4\rangle\right) = \frac{1}{\sqrt{2}}\sqrt{4}|3\rangle = \sqrt{2}|3\rangle \tag{2}$$

$$a|\bar{1}\rangle = \sqrt{2}|1\rangle \tag{3}$$

are supported on odd Fock states $1, 3$ while codewords are supported on even Fock states $0, 2, 4$. Notice that PaP consists of overlaps between codewords and error words,

$$PaP = \sum_{\mu,\nu \in \mathsf{Z}_2} |\bar{\mu}\rangle\langle\bar{\mu}|a|\bar{\nu}\rangle\langle\bar{\nu}| \,, \tag{4}$$

implying that PaP must be zero because the two respective sets are supported on different Fock states.
The remaining condition $Pa^\dagger a P$ is easily determined. Notice first that $a^\dagger a$ does not cause transitions between Fock states, so $\langle\bar{1}|a^\dagger a|\bar{0}\rangle = 0$. We are left with

$$\langle\bar{0}|a^\dagger a|\bar{0}\rangle = 2 \tag{5}$$

$$\langle\bar{1}|a^\dagger a|\bar{1}\rangle = 0 + \frac{4}{2} = 2 \tag{6}$$

which proves what we wanted.

(b): The error set is now $\{B_0, B_1, B_2, B_3\}$. Similar logic applies for $PB_i^\dagger B_j P$ with $i \neq j$. Notice that $B_{i>0} \propto a_i$, the loss operator on the ith mode, while B_0 does change the Fock states supporting a state. This is all we need to know to determine the Fock state support of error words $B_i|\bar{\mu}\rangle$. For logical zero, $\mu = 0$,

$$B_0|\bar{0}\rangle \propto |111\rangle \tag{7}$$
$$B_1|\bar{0}\rangle \propto |011\rangle \tag{8}$$
$$B_2|\bar{0}\rangle \propto |101\rangle \tag{9}$$
$$B_3|\bar{0}\rangle \propto |110\rangle \tag{10}$$

and for logical one, $\mu = 1$,

$$B_0|\bar{1}\rangle \in \text{span}\left\{|300\rangle, |030\rangle, |003\rangle\right\} \tag{11}$$
$$B_1|\bar{1}\rangle \propto |200\rangle \tag{12}$$
$$B_2|\bar{1}\rangle \propto |020\rangle \tag{13}$$
$$B_3|\bar{1}\rangle \propto |002\rangle \tag{14}$$

All eight of these words are supported on different Fock states, so $PB_i^\dagger B_j P = 0$ for all $i \neq j$.
We are left with calculating the $i = j$ case, which again only needs diagonal overlaps $\langle\bar{\mu}|B_i^\dagger B_i|\bar{\mu}\rangle$ since $B_i^\dagger B_i$ is function of photon number operators $a_i^\dagger a_i$ only, and such operators do not raise or lower Fock states. The $i = 0$ case is

$$\langle\bar{0}|B_0^\dagger B_0|\bar{0}\rangle = (1-\gamma)^3 \tag{15}$$
$$\langle\bar{1}|B_0^\dagger B_0|\bar{1}\rangle = (1-\gamma)^3 \,. \tag{16}$$

Notice that the codewords are permutation invariant, so $\langle\bar{\mu}|B_i^\dagger B_i|\bar{\mu}\rangle$ for $i \neq 0$ are all equal. Thus we only need to calculate

$$\langle\bar{0}|B_1^\dagger B_1|\bar{0}\rangle = \gamma\left(1-\gamma\right)^2 \tag{17}$$

$$\langle\bar{1}|B_1^\dagger B_1|\bar{1}\rangle = \gamma\left(1-\gamma\right)^2 . \tag{18}$$

(c): Codewords for a code with total photon number 1 would be in the single-excitation manifold, i.e.,

$$\text{span}\left\{|100\cdots\rangle, |010\cdots\rangle, \cdots, |\cdots001\rangle\right\} . \tag{19}$$

For each basis vector in the list above, there exists an error B_i that maps it to the vacuum state $|00\cdots0\rangle$. Therefore, there exists an $i, j > 0$ such that two error words overlap, i.e.,

$$\langle\bar{1}|B_i^\dagger B_j|\bar{0}\rangle \neq 0 . \tag{20}$$

In other words, there are no two-dimensional error spaces for this combination of code and error set.

Problem 2

(a): These are distinct, so we need to determine whether these generate an abelian group that does not contain $-I$.
Any two of them have different non-identity Paulis only at two locations, meaning that they commute.
To check if these generate $-I$, we iterate over all powers of their products. These are the single generators, products of any two of them, the product of all three of them, and their squares.

- The single generators are distinct from $-I$ and square to I.
- Products of two generators are not equal to $-I$, but can square to $-I$ whenever there are an odd number of locations where the two generators have different non-identity Paulis. (For example, for single-qubit generators X, Z, the square of the product $(XZ)^2 = XZXZ = -I$.) However, we have already established that the generators in question have different non-identity Paulis only at an even number of locations, so squares of their products will all be $+I$.
- The triple product is

$$(XXZYI)(ZYIIX)(XIXZZ) = (XZX)(XYI)(ZIX)(YIZ)(IXZ) \tag{21}$$

$$= (-Z)(iZ)(iY)(iX)(-iY) \tag{22}$$

$$= ZZYXY \tag{23}$$

which doesn't equal $-I$ and squares to $+I$.

Therefore, this is a set of stabilizer generators. These define a $[[5,2,2]]$ code because of the following:

- To determine number of logical qubits, subtract number of generators from n, $k = n - r = 5 - 3 = 2$.
- To determine distance, we check which Pauli strings commute with all stabilizers but are not in the stabilizer group. Single-qubit Paulis anticommute with at least one stabilizer generator. The two-qubit Pauli string $XXIII$ commutes with all of them and is not in the stabilizer group, so $d = 2$.

(b): Not a valid stabilizer group since $X^{\otimes 6}Y^{\otimes 6}Z^{\otimes 6} = (XYZ)^{\otimes 6} = i^6 I = -I$.

(c): First half of first row, 001101, shares a 1 at only one location with the second half of the second row, 011010. This means the corresponding stabilizers do not commute, so these do not generate a stabilizer group.

(d): Vectors v and ωv in the classical $GF(4)$ code are considered linearly dependent, but they convert to independent Paulis (see Gottesman's book, pg. 79-80), so the generators corresponding to this are $IXXZY$ and $IZZYX$. These are clearly distinct, so we need to determine whether these generate an abelian group that does not contain $-I$.
They have different non-identity Paulis at four locations, meaning that they commute.
They square to $+I$, and their product

$$(IXXZY)(IZZYX) = I(iY)(iY)(-iX)(-iZ) = IYYXZ \tag{24}$$

is neither $-I$ nor squares to it. These generate a valid stabilizer group.
There is no protection on the first qubit since both generators are identity there. This is a $[[5,3,1]]$ code.

Problem 3

A stabilizer code is degenerate if there is a non-identity stabilizer group element whose weight is less than the distance (Def. 3.9 in Gottesman's book). The generators of an asymptotically good LDPC code family have weights that are independent of n. Given some generator of weight c, there exists a code family member $[[n,k,d]]$ with sufficiently large distance such that errors of weight $\leq c$ are correctable. Therefore, this code and all larger codes in this family are degenerate.

Problem 4

**(a):** Plugging $[[5,1,3]]$ into the bound yields

$$\sum_{s=0}^{t} \binom{n}{s}(p^2-1)^s = \sum_{s=0}^{1}\binom{5}{s}(p^2-1)^s = 1 + 5(p^2-1) \leq p^4 \tag{25}$$

To determine the p at which this is an equality, we can solve a quadratic equation for p^2. The positive solutions are $p = 1$ and $p = 2$, with the latter qubit case being the only one applicable to quantum codes.

(b): We apply the strategy from 4(a), yielding the inequality

$$36p^4 - 63p^2 + 28 \leq p^8 . \tag{26}$$

Let's bring everything to one side and plot the resulting polynomial on that side:

```
Plot[x⁸ - 36 x⁴ + 63 x² - 28, {x, 0, 2.5}]
```
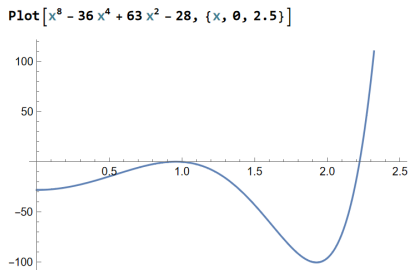

Notice that there are crossings through the horizontal axis at integer values of x, so the inequality is not saturated for any relevant p. The polynomial blows up to ∞ as $x \to \infty$, so this is an inequality for all $p > 2$. It dips below the horizontal axis at $p = 2$, meaning that the inequality is violated there.

(c): Plugging in the parameters as before yields the inequality

$$1 + 8n \leq 3^{n-k} . \tag{27}$$

This is an equality for $n = 10$ and $k = 6$ code, which can be obtained by brute force:

```
Table[{n, n - Log[3, 8 n + 1] // N}, {n, 1, 15}] // Grid
```

```
1     -1.
2   -0.578902
3    0.070053
4    0.817342
5    1.61976
6    2.45751
7    3.31986
8    4.20031
9    5.09466
10      6.
11   6.91427
12   7.83592
13   8.76378
14   9.69695
15   10.6347
```

Problem 5

(a): This is a $[[4,2,2]]_3$ code since every weight-one Pauli doesn't commute with at least one generator, but the Pauli string $X^\dagger XII$ commutes with both generators and is not in the stabilizer group.

(b): Consider the set

$$\overline{X}_1 = X^\dagger X I I \tag{28}$$

$$\overline{X}_2 = I I X^\dagger X \tag{29}$$

$$\overline{Z}_1 = Z Z^\dagger I I \tag{30}$$

$$\overline{Z}_2 = I I Z Z^\dagger . \tag{31}$$

These satisfy the Pauli group relations, namely

$$\overline{Z}_i \overline{X}_j = \omega^{\delta_{ij}} \overline{X}_j \overline{Z}_i . \tag{32}$$

The distinct support ensures this is true for $i \neq j$. For the diagonal $i = j = 1$ case,

$$\overline{Z}_1 \overline{X}_1 = \left(Z X^\dagger \right) \left(Z^\dagger X \right) I I = \left(\omega^\star X^\dagger Z \right) \left(\omega^\star X Z^\dagger \right) I I = \omega^{-2} \overline{X}_1 \overline{Z}_1 = \omega \overline{X}_1 \overline{Z}_1 , \tag{33}$$

and similarly for $i = 2$.

(c): We can try to construct the $|\overline{00}\rangle$ state by applying the codespace projector onto the $|0000\rangle$ state,

$$\Pi|0000\rangle = \frac{1}{9} \left[I + X X Z Z + Z Z X X + X^2 X^2 Z^2 Z^2 + Z^2 Z^2 X^2 X^2 + (X Z)(X Z)(Z X)(Z X) + \cdots \right] |0000\rangle \tag{34}$$

$$\propto |0000\rangle + |1100\rangle + |0011\rangle + |2200\rangle + |0022\rangle + \omega^2 |1111\rangle + \omega|1122\rangle + \omega|2211\rangle + \omega^2 |2222\rangle . \tag{35}$$

This is in the codespace, so it is a +1 eigenstate of all stabilizers. Let's check if this is indeed a +1 eigenstate of $\overline{Z}_{i \in \{1,2\}}$. This is the case since each basis element in the expansion consists of identical values in the 1st and 2nd qutrits as well as in the 3rd and 4th, and since for such pairs, $Z Z^\dagger |nn\rangle = \omega\omega^\star |nn\rangle = |nn\rangle$). Therefore, the normalized version of the above vector is $|\overline{00}\rangle$.