Problem Set #2

Classical and Quantum Codes
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Problem #1. Bosonic codes (15 points)

A bosonic mode is an infinite-dimensional Hilbert space with a standard basis labelled by the non-negative integers, i.e., $|0\rangle, |1\rangle, |2\rangle, |3\rangle, \ldots$, representing the eigenstates of a harmonic oscillator. For instance, for light, $|j\rangle$ is a state with j photons in this mode (the mode specifying a particular wavenumber and spatial profile). The harmonic oscillator comes with a set of raising and lowering operators, $a|n\rangle = \sqrt{n}|n-1\rangle$ and $a^\dagger$ satisfying $[a, a^\dagger] = 1$, as well as the number operator $a^\dagger a|n\rangle = n|n\rangle$.

For bosonic modes, there is a natural generalization of the amplitude damping channel to

$$\rho \mapsto \sum_{k=0}^{\infty} A_k \rho A_k^\dagger, \tag{1}$$

with

$$A_k = \left(\frac{\gamma}{1-\gamma}\right)^{k/2} \frac{a^k}{\sqrt{k!}} (1-\gamma)^{a^\dagger a/2} = \sum_{j \geq k} \sqrt{\binom{j}{k}} \sqrt{(1-\gamma)^{j-k}\gamma^k}\ |j-k\rangle\langle j|, \tag{2}$$

representing loss of k photons from a mode. γ indicates the rate of photon loss. In particular,

$$A_0 = \sum_j (1-\gamma)^{j/2}\ |j\rangle\langle j| \tag{3}$$

$$A_1 = \sum_{j\geq 1} \sqrt{j(1-\gamma)^{j-1}\gamma}\ |j-1\rangle\langle j|. \tag{4}$$

Note that, as with amplitude damping, A_0 is not proportional to the identity — more highly excited states are more likely to emit photons, so not having a photon loss event makes it more likely there were fewer photons to begin with.

For this problem, we will look at codes encoding a single qubit in n bosonic modes to correct for loss of a single photon from one mode. Let $B_0 = A_0^{\otimes n}$ be the no-loss operator and $B_i = A_0^{\otimes i-1} \otimes A_1 \otimes A_0^{\otimes n-i}$ be the operator which has loss of 1 photon from the ith mode and no loss from the other modes. The error set that we are trying to correct is thus $\mathcal{E} = \{B_0, B_1, \ldots B_n\}$ or, for the first part, its approximation to lowest order in γ.

a) Consider the following encoding:

$$|\overline{0}\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |4\rangle) \tag{5}$$

$$|\overline{1}\rangle = |2\rangle. \tag{6}$$

Show that this is a QECC correcting the error set $\{I, a\}$ for one mode.

b) Consider the following encoding:

$$|\bar{0}\rangle = \frac{1}{\sqrt{3}}(|300\rangle + |030\rangle + |003\rangle) \tag{7}$$

$$|\bar{1}\rangle = |111\rangle. \tag{8}$$

Show that this is a QECC correcting the error set $\mathcal{E}$ for three modes.

c) The total photon number of a multimode basis state $|j_1 j_2 \dots j_n\rangle$ is $\sum_i j_i$. The total photon number of a superposition is only defined if all terms in the superposition have the same total photon number, and is then equal to that value. Thus, the codewords for the code in part b have total photon number 3. Show that there is no bosonic code correcting $\mathcal{E}$ for any number of modes that has total photon number 1.

Problem #2. Example stabilizer (20 points)

For each of the following sets of Paulis, determine if they define valid stabilizers. If so, give their parameters $[[n, k, d]]$.

a) Stabilizer is all products of these operators:

$$\begin{matrix} X & X & Z & Y & I \\ Z & Y & I & I & X \\ X & I & X & Z & Z \end{matrix}$$

b) Stabilizer is all products of these operators:

$$\begin{matrix} X & X & X & X & X & X \\ Y & Y & Y & Y & Y & Y \\ Z & Z & Z & Z & Z & Z \end{matrix}$$

c) In binary symplectic matrix form:

$$\left(\begin{array}{cccccc|cccccc} 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 & 1 \end{array}\right)$$

d) The stabilizer corresponding to the GF(4) linear code with the following parity check matrix:

$$\begin{pmatrix} 0 & 1 & 1 & \omega & \omega^2 \end{pmatrix}$$

Problem #3. Low-density parity check CSS codes (10 points)

A classical LDPC ("low density parity check") code is an $[n, k, d]$ linear code where each row of the parity check matrix has at most r 1's and each column of the parity check matrix has at most c 1's, with r and c of constant size (as n gets large). (Sometimes LDPC codes with r and c increasing sublinearly with n are also considered, but assume r and c are constant for the purposes of this problem.) Classical LDPC codes are interesting because they can achieve good values of k/n, d/n, and also generally have good decoding algorithms.

A quantum LDPC code is a stabilizer code for which each generator has low weight and each qubit appears in only a small number of generators. One might try to make good quantum LDPC codes using the CSS construction, based on pairs of classical LDPC codes $C_1(n)$ and $C_2(n)$. Suppose that one finds a family of such codes which produce $[[n, k, d]]$ quantum codes with k/n and d/n both constant as n gets large. Show that this family of quantum codes must be degenerate for large n.

[No such family is known in the quantum case. UPDATE: See arXiv:2111.03654. The point of the problem is that, because degeneracy is important to find such codes, the quantum case is not a straightforward application of the CSS construction.]

Problem #4. Quantum Hamming bound for qudit codes (15 points)

The quantum Hamming bound for qudits of dimension p becomes

$$\sum_{s=0}^{t} \binom{n}{s} (p^2 - 1)^s \leq p^{n-k}, \tag{9}$$

which must hold for non-degenerate $((n, p^k, 2t+1))_p$ codes.

a) For what values of p does a $[[5, 1, 3]]_p$ code saturate the quantum Hamming bound?

b) For what values of p would a $[[9, 1, 5]]_p$ code saturate the quantum Hamming bound? For which values of p would the code violate the quantum Hamming bound? (Note that such a code is only known to exist for prime power p with $p \geq 9$.)

c) For $p = 3$, find the smallest integer values of n and k such that an $[[n, k, 3]]_3$ code saturates the quantum Hamming bound or show that no integer n and k work.

Problem #5. Logical operations for qudit code (15 points)

Consider the following stabilizer code for qutrits (qudits with dimension $p = 3$):

$$\begin{matrix} X & X & Z & Z \\ Z & Z & X & X \end{matrix}$$

a) What are its parameters as a QECC?

b) Find a generating set for the logical Pauli group. (I.e., coset representatives for $\overline{X}_i$ and $\overline{Z}_i$).

c) For your choice of logical Pauli operators, write down the codeword with all logical qubits 0 expanded in the standard basis for the physical qubits.