ENEE626. Problem set 1. Due in class on 9/24/15.

1. Write out the parameters $[n, k, d]$, a generator and a parity-check matrix for the binary linear codes $\mathcal{C}_1 = \{0^n\}$ (one vector), $\mathcal{C}_2 = F$ (the entire space), $\mathcal{C}_3 = \{0^n, 1^n\}$ (the repetition code), $\mathcal{C}_4 = \{\text{all even-weight vectors}\}$ (the single parity-check code).

2. Consider a binary linear code $\mathcal{C}$ spanned by the rows of the matrix

$$\begin{matrix} 000110001 \\ 110001000 \\ 111011110 \end{matrix}$$

(a) Find the parameters (length, dimension, and distance) of the code $\mathcal{C}$.

(b) Represent the code in a systematic form so that the message bits appear in the coordinates 2,3, and 5 of the codeword.

(c) Find a parity-check matrix H of the code $\mathcal{C}$.

(d) Find the parameters of the code $\mathcal{C}^\perp$ spanned by the rows of H.

3. Consider a binary linear code $\mathcal{C}$ with generator matrix

$$\begin{matrix} 101010 \\ 010011 \\ 100101 \end{matrix}$$

Write out the standard array for the code $\mathcal{C}$. Identify all correctable and non-correctable errors (explain your conclusions). What is the distance of $\mathcal{C}$?

4. Let G and H be a generator and a parity-check matrix of a binary code $\mathcal{C}$ of length n and dimension k. Let $E \subset \{1, 2, \dots, n\}, |E| \le k$. Recall that $G(E)$ denotes the submatrix of G formed of the columns indexed by E.

(a) Prove that if $\text{rk}(G(E)) = |E|$ then $\text{rk}(H(E^c)) = n - k$ (i.e., that E^c contains a check set of $\mathcal{C}$).

(b) Prove that $k - \text{rk}(G(E)) = |E^c| - \text{rk}(H(E^c))$.

5. Let $D = \mathcal{H}_4$ be the binary Hamming code of length 15, i.e., the code whose parity-check matrix is formed all the 15 nonzero columns of length 4, taken in the lexicographic order (from 0001 to 1111).

(a) Find the dimension k and distance d of D (explain your answers).

(b) Write out a generator matrix of D such that the message bits are bits $1, 2, \dots, k$.

(c) You are given a received vector $z = 0000000 * 0000111$ where $*$ stands for erasure. Perform maximum likelihood decoding of z with the code D. What is/are the candidate codeword(s)? Explain your answer.

6. Binomial coefficients. Below all the parameters are whole numbers. Prove that

(i) $\binom{n}{k} = \binom{n}{n-k}$;

(ii) $\binom{n-1}{k} + \binom{n-1}{k-1} = \binom{n}{k}$;

(iii) $(n-k)\binom{n}{k} = n\binom{n-1}{k}$;

(iv) $\sum_{k \text{ even}} \binom{n}{k} = \sum_{k \text{ odd}} \binom{n}{k} = 2^{n-1}$;

(v) $\sum_k (-1)^k \binom{n}{k} = 0$;

(vi) $\sum_{0 \le k \le n} \binom{k}{m} = \binom{n+1}{m+1}$;

(vii) $\binom{r}{j}\binom{j}{s} = \binom{r}{s}\binom{r-s}{j-s}$;

(viii) $\sum_{j=0}^{(r-1)/2} (-1)^j \binom{r}{j}(r-2j) = 0$ (r odd);

(viii) $\sum_i (-1)^i \binom{p}{i}\binom{i}{s} = (-1)^p \delta_{p,s}$

(ix) $\sum_k \binom{r}{m+k}\binom{s}{n-k} = \binom{r+s}{m+n}$.

(x) $\sum_i i\binom{n}{i} = n2^{n-1}$

(xi) $\sum_i i^2\binom{n}{i} = \frac{n(n+1)}{4}2^n$.

(xii) $\sum_{i=0}^{m} (-1)^i \binom{n}{i} = (-1)^m \binom{n-1}{m}$.