ENEE620. Midterm examination 2, 11/14/2024. Instructor: A. Barg

- Please submit your work to ELMS Assignments as a single PDF file. You must submit your paper within 3 hours from accessing the exam paper online.
- The exam has **5 problems**. Each problem is 10 points. Max score=50 points
- Your answers should be justified. Giving just the answer may result in no credit for the problem.
- Please pay attention to the writing. You may lose points if your paper is difficult to read.

Problem 1. (RANDOM WALK WITH REFLECTING SCREEN) Consider a symmetric random walk on $\mathbb{N}_0 = \{0, 1, 2, \dots\}$ that moves $i \to i-1$ (left) or $i \to i+1$ (right) with equal probabilities, except for $i = 0$, where $p_{01} = 1$.

(a) What is the generating function of the return distribution to the origin? In other words, let

$$f_0(n) = P(X_n = 0 | X_{n-1} \neq 0, \dots, X_1 \neq 0, X_0 = 0), \quad n \geq 1.$$

Your task is to find a closed-form expression (thus, no sums in the answer) for $F_0(z) = \sum_{n \geq 1} f_0(n) z^n$ (Hint: Lecture notes should help).

(b) Using $F_0(z)$ that you found in part (a), argue whether the process is recurrent or transient.

(c) Using $F_0(z)$ that you found in part (a), argue whether the process is null recurrent or not.

(d) Let P denote the (semi-infinite) transition matrix of the process. Is there a stationary vector π, i.e., an eigenvector satisfying $\pi P = \pi$? If yes, express all $\pi_i, i \geq 1$ via π_0.

(e) Does this process (Markov chain) have a limiting distribution? Justify your answer.

SOLUTION: (a) In Lecture 18 we found the generating function for the probability of hitting 1 in a symmetric random walk. This generating function is inherited by the current process (the reflective random walk) because from 0 we move to 1 with probability 1, and then it becomes the question of hitting 1 (acutally, -1, but the picture is symmetric). Lifting the generating function from Lecture 18, we have

$$G_0(z) = \frac{1 - (1 - z^2)^{1/2}}{z} = \sum_{i=1}^{\infty} g(i) z^i,$$

where $g(i)$ is the probability of hitting 0 in i steps starting at 1. Our generating function equals (noting that $f(0) = 0$

$$\begin{aligned} F_0(z) &= \sum_{i=1}^{\infty} f_0(n) z^n = \sum_{i=1}^{\infty} g(i-1) z^i = z \sum_{i=0}^{\infty} f(i) z^i \\ &= 1 - \sqrt{1 - z^2}. \end{aligned}$$

(b) The probability of ever returning to 0 equals $F_0(1) = 1$, so the origin, and thus the entire process, is recurrent.

(c) At the same time, the expected return time to 0 is $F_0'(1) = \frac{z}{1-z^2}\Big|_{z=1} = \infty$, and thus the process is null recurrent.

(d) Writing out the equations for π_n from $\pi = \pi P$, we find $\pi_n = 2\pi_0$ for all $n \geq 1$, and thus any vector of the form $\pi = (a, 2a, 2a, \dots)$, where a can be any real number including 0, is an eigenvector of P.

(e) For one thing, the vector π cannot be normalized to a pmf irrespective of a, so there is no limiting distribution. Independently, the process is null recurrent, and so the limiting distribution does not exist.

Problem 2.

Let $(X_n)_n$ be a sequence of independent Gaussian random variables with $X \sim \mathcal{N}(0, 1)$ and let $S_n = X_1 + \cdots + X_n, n \geq 1$. Prove that the sequence $Y_n = \frac{1}{\sqrt{n+1}} \exp\left\{\frac{S_n^2}{2(n+1)}\right\}, n \geq 1$ is a martingale with respect to the filtration $(\mathcal{F}_n = \sigma(X_1, \dots, X_n))_{n \geq 1}$.

SOLUTION:

(a) We have

$$E|Y_n| = EY_n = \frac{1}{\sqrt{n+1}} E\Big[\exp\frac{(\sum X_i)^2}{2(n+1)}\Big] = \frac{1}{\sqrt{n+1}} E\Big[\exp\frac{\sum_i X_i^2 + 2\sum_{i<j} X_i X_j}{2(n+1)}\Big]$$
$$= \frac{1}{\sqrt{n+1}} E\prod_{i=1}^n e^{\frac{X_i^2}{(2(n+1))^{1/n}}} = \frac{1}{\sqrt{n+1}} \prod_{i=1}^n E e^{\frac{X_i^2}{(2(n+1))^{1/n}}}$$

Each of the expectations under the product is finite, and so is the product:

$$Ee^{\frac{X^2}{2a}} = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{\frac{x^2}{2a} - \frac{x^2}{2}} dx = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{-\frac{x^2}{2/(1-a)}} dx = \frac{1}{\sqrt{1-a}}.$$

Now let us compute $E(Y_{n+1}|\mathcal{F}_n)$. We have

$$Y_{n+1} = \frac{1}{\sqrt{n+2}} \exp\Big(\frac{S_n^2 + 2X_{n+1}S_n + X_{n+1}^2}{2(n+2)}\Big)$$

and

$$E(Y_{n+1}|\mathcal{F}_n) = \frac{1}{\sqrt{n+2}} E\Big[\exp\Big(\frac{S_n^2 + 2X_{n+1}S_n + X_{n+1}^2}{2(n+2)}|\mathcal{F}_n\Big)\Big]$$
$$= \frac{1}{\sqrt{n+2}} \exp\Big(\frac{S_n^2}{2(n+2)}\Big) E\Big[\exp\frac{2X_{n+1}S_n + X_{n+1}^2}{2(n+2)}\Big)|\mathcal{F}_n\Big]$$
$$= \frac{1}{\sqrt{n+2}} e^{\frac{S_n^2}{2(n+2)}} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\Big(\frac{2xS_n + x^2}{2(n+2)} - \frac{x^2}{2}\Big) dx$$
$$= \frac{1}{\sqrt{n+2}} e^{\frac{S_n^2}{2(n+2)}} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\Big(\frac{2xS_n - (n+1)^2}{2(n+2)}\Big) dx$$
$$= \frac{1}{\sqrt{n+2}} e^{\frac{S_n^2}{2(n+2)}} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\Big(\frac{n+1}{2(n+2)}\Big(-x^2 + \frac{2S_n}{n+1}x - \frac{S_n^2}{(n+1)^2} + \frac{S_n^2}{(n+1)^2}\Big) dx$$
$$= \frac{1}{\sqrt{n+2}} e^{\frac{S_n^2}{2(n+2)} + \frac{S_n^2}{2(n+1)(n+2)}} \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \exp\Big[-\frac{(x - \frac{S_n^2}{n+1})^2}{2\frac{n+2}{n+1}}\Big] dx$$
$$= \frac{1}{\sqrt{n+2}} e^{\frac{S_n^2}{2(n+1)}} \sqrt{\frac{n+2}{n+1}} \frac{1}{\sqrt{2\pi\frac{n+2}{n+1}}} \int_{-\infty}^{\infty} \exp\Big[-\frac{(x - \frac{S_n^2}{n+1})^2}{2\frac{n+2}{n+1}}\Big] dx$$
$$= \frac{1}{\sqrt{n+1}} e^{\frac{S_n^2}{2(n+1)}} = Y_n,$$

as required.

Problem 3. Consider a sequence $(X_n)_n$ of RVs on $(\Omega, \mathcal{F}, P)$ defined as

$$X_n(\omega) = \begin{cases} n & \text{if } \omega \in A_n \\ 0 & \text{if } \omega \in A_n^c, \end{cases}$$

where $(A_n)_n$ is a sequence of events with $P(A_n) = \frac{1}{n}$ for each n.

(a) Determine whether $X_n \overset{\text{a.s.}}{\to} 0$.

(b) Determine whether $X_n \overset{\text{p}}{\to} 0$.

(c) Determine whether $X_n \to 0$ in L_1, i.e., whether $E|X_n| \to 0$ as $n \to \infty$.

(d) Find the distribution function $F_{X_n}(x)$ for all n. Does the sequence $(X_n)_n$ converge to 0 in distribution?

(e) Is the sequence $(X_n)_n$ uniformly integrable?

SOLUTION: (a) If the events A_n are mutually independent, then $P(A_n \text{i.o.}) = 1$, and the sequence X_n does not converge to 0 a.s.

(b) $P(X_n \leq 0) = \frac{1}{n} \to 0$ so $X_n \overset{\text{p}}{\to} 0$.

(c) Since $EX_n = 1$, it does not converge to 0 with $n \to \infty$.

(d) We have

$$F_{X_n}(x) = \begin{cases} 0 & \text{if } x < 0 \\ \frac{n-1}{n} & \text{if } 0 \le x < n \\ 1 & \text{if } x \ge n. \end{cases}$$

The target limiting distribution is a step at 0 (known as the Heaviside step $H(x) := \mathbb{1}_{\{x \ge 0\}}$), and $F_{X_n}(x) \to H(x)$ for all $x \neq 0$, i.e., at all the continuity points of $H(x)$. In conclusion, $X_n \xrightarrow{\text{d}} 0$.

(e) Checking the definition, we need that $\lim_{M\to\infty} \sup_n E(X_n \mathbb{1}_{|X_n| \ge M}) = 0$. However, for any M, $\sup_n E(X_n \mathbb{1}_{|X_n| \ge M}) = 1$, so the expression $\sup_n E(X_n \mathbb{1}_{|X_n| \ge M})$ as a function of M does not approach 0 for increasing M. As a result, the sequence $(X_n)_n$ is not uniformly integrable.

Problem 4. Suppose that $(X_n)_{\{n \ge 0\}}$ is an irreducible and aperiodic Markov chain with a finite state space $S = \{1, 2, \ldots, N\}$, transition matrix P, and stationary distribution $\pi = (\pi_1, \ldots, \pi_N)$. Suppose also that (X_n) is reversible.

(a) Show that for any function $f : S \to \mathbb{R}$

$$E(f(X_n)f(X_{n+1})) \overset{n\to\infty}{\longrightarrow} \sum_{i \in S} \pi_i P_{ii} f(i)^2 + 2 \sum_{i<j} \pi_i P_{ij} f(i) f(j).$$

(b) Suppose in addition that $\Pr(X_0 = j) = \pi_j, j \in S$ (the chain starts at stationarity). Show that the covariance

$$\text{Cov}(f(X_0), f(X_n)) = \sum_{i,j} \pi_i P_{ij}^{(n)} f(i) f(j) - (E_\pi(f))^2,$$

where $E_\pi(f) := \sum_{i\in S} \pi_i f(i)$.

(c) With the assumptions as in part (b), show the correlation decline, namely that

$$\lim_{n\to\infty} \text{Cov}(f(X_0), f(X_n)) = 0.$$

SOLUTION: (a) We have

$$E(f(X_n)f(X_{n+1})) = \sum_{i,j} \Pr(X_n = i, X_{n+1} = j) f(i) f(j) = \sum_{i,j} P(X_n = i) P_{ij} f(i) f(j)$$
$$= \sum_{i=1}^{N} P(X_n = i) P_{ii} f(i)^2 + \sum_{i \neq j} P(X_n = i) P_{ij} f(i) f(j).$$

Taking the limit for $n \to \infty$ on the right, we have $P(X_n = j) \to \pi_j$ by the main ergodic theorem for MCs, and

$$\sum_{i\neq j} P(X_n = i) P_{ij} f(i) f(j) \to \sum_{i\neq j} \pi_i P_{ij} f(i) f(j) = 2 \sum_{i<j} \pi_i P_{ij} f(i) f(j),$$

where in the second equality we used reversibility, $\pi_i P_{ij} = \pi_j P_{ji}$. Substituting, we obtain the desired formula.

(b) First,

$$\text{Cov}(f(X_0), f(X_n)) = E(f(X_0)f(X_n)) - E(f(X_0))E(f(X_n)).$$

As above,

$$E(f(X_0)f(X_n)) = \sum_{i,j} \pi_i P_{ij}^{(n)} f(i) f(j)$$

Plainly, $E(f(X_0)) = E(f(X_n)) = \sum_{i\in S} \pi_i f(i) = E_\pi(f)$.

(c) In the limit, $P_{ij}^{(n)} \to \pi_j$ for all i, and so

$$\sum_{i,j} \pi_i P_{ij}^n f(i) f(j) \to \sum_i \pi_i f(i) \sum_j \pi_j f(j) = (E_\pi(f))^2,$$

showing that the formula in part (b) approaches 0 as n increases.

Problem 5. We are given a discrete RV $X \geq 0$ and a Borel function $g : \mathbb{R}_+ \to \mathbb{R}_+$. Suppose that $EX < \infty$.

(a) Does $E(g(X))$ exist, and if yes, how do you compute it? Justify your answers.

(b) Let $g(x) = \frac{1}{1+x}$. Find $E(g(X))$ (in closed form, no finite or infinite sums) if

(1) $X \sim \text{Bin}(n, p)$ is a binomial RV, where $n \geq 1$ and $0 < p < 1$;

(2) $X \sim \text{Geo}(p)$ is geometric, with $0 < p < 1$;

(3) $X \sim \text{Poi}(\lambda)$ is Poisson, with $\lambda > 0$.

SOLUTION: (a) By the change of measure argument (Lec.7)

$$\int_\Omega f(X(\omega))dP(\omega) = \int_{-\infty}^{\infty} g(x)dF_X(x).$$

In the context of this question, $E(g(x)) = \sum_x g(x)P(X = x)$, where the sum extends to the range of X. This sum may be finite or infinite depending on the pmf of X. In all the examples below, it is finite.

(b) (1) We have $X \sim \text{Bin}(n, p)$ and

$$\begin{aligned}
E\Big(\frac{1}{1+X}\Big) &= \sum_{i=0}^n \frac{1}{i+1}\binom{n}{i}p^i(1-p)^{n-i} = \frac{1}{n+1}\sum_{i=0}^n \frac{n+1}{i+1}\binom{n}{i}p^i(1-p)^{n-i} \\
&= \frac{1}{p(n+1)}\sum_{i=0}^n \binom{n+1}{i+1}p^{i+1}(1-p)^{(n+1)-(i+1)} \\
&= \frac{1}{p(n+1)}\sum_{i=1}^{n+1} \binom{n+1}{i}p^{i}(1-p)^{n+1-i} \\
&= \frac{1}{p(n+1)}((n+1)p - (1-p)^{n+1}) = \frac{1-(1-p)^{n+1}}{p(n+1)}.
\end{aligned}$$

(2) We use the definition of geometric RV as the wait time till the first H in a sequence of $(p, 1-p)$ coin tosses, not including the H itself. Thus,

$$P(X = n) = p(1-p)^n, n \geq 0.$$

For the expectation we obtain

$$E\Big(\frac{1}{1+X}\Big) = p\sum_{n=0}^\infty \frac{(1-p)^n}{n+1} = \frac{p}{1-p}\sum_{n=1}^\infty \frac{(1-p)^n}{p} = \frac{p}{1-p}\ln\frac{1}{p}$$

since $\ln(1-x) = -\sum_{n=1}^\infty \frac{x^n}{n}$, so the sum equals $-\ln(1-(1-p)) = \ln\frac{1}{p}$.

We could also use the definition that includes the toss H in the value, and then $P(X = n) = p(1-p)^{n-1}, n \geq 1$, resulting in a similar calculation.

(3)

$$\begin{aligned}
E\Big(\frac{1}{1+X}\Big) &= \sum_{i=0}^\infty \frac{1}{1+i}\frac{\lambda^i}{i!}e^{-\lambda} = \frac{e^{-\lambda}}{\lambda}\sum_{i=0}^\infty \frac{\lambda^{i+1}}{(i+1)!} \\
&= \frac{e^{-\lambda}}{\lambda}(e^\lambda - 1) = \frac{1-e^{-\lambda}}{\lambda}.
\end{aligned}$$