ENEE620. Midterm examination 2, 11/19/2020. Instructor: A. Barg

- Please submit your work as a single PDF file to ELMS/Canvas Assignments by 11/19, 4pm US Eastern time.
- The exam paper consists 5 problems, each is worth 10 points. Max score=50 points
- Your answers should be justified. Giving just the answer may result in no credit for the problem.
- Please pay attention to the writing. You may lose points if your paper is difficult to read.

Problem 1. Let U be an RV on a probability space $(\Omega, \mathcal{F}, P)$. Define the RVs X, Y, Z as follows:

$$X = \sin(U), \quad Y = \frac{U}{1+U^2}, \text{ and } Z = U\cos(U).$$

(a) Determine whether the expectations EX, EY, EZ exist and if so, whether they are finite (with no additional assumptions on the distribution of U). Justify your answers!

For the remaining questions in this problem assume that the RV U is symmetric in the sense that U and $-U$ have the same probability distribution.

(b) Find EX, EY.

(c) Give an example to show that EZ may not exist. Find an additional condition on U under which EZ always exists and find the value of EZ. (Hint: one way to proceed is to use that, by definition, the expectation of an RV ξ equals $E\xi = E\xi^+ - E\xi^-$, where $\xi^+ = \max(\xi, 0), \xi^- = -\min(\xi, 0)$.)

Solution:

(a) $|X| \le 1$ and $|Y| \le 1$, so EX and EY exist. The expectation EZ may not exist, depending on the distribution of U.

(b) Since U is symmetric, we have $\sin(-U) \overset{d}{=} \sin U$, and so $E(-X) = -E(X) = E(X)$, implying that $EX = 0$. Similarly, $\frac{-U}{1+(-U)^2} \overset{d}{=} \frac{U}{1+U^2}$, leading us to conclude that $EY = 0$.

(c) We can obtain infinite expectation by taking a Cauchy-like pmf on $\mathbb{Z}$. Specifically, let $p_U(2\pi n) = \frac{C}{1+n^2}$ where C is the normalizing constant. Take the expectation of $Z^+ = \max(Z, 0)$ and $Z^- = -\min(Z, 0)$:

$$EZ^{\pm} = C\sum_{n=1}^{\infty} 2\pi n \frac{\cos(2\pi n)}{1+n^2} = 2\pi C \sum_{n=1}^{\infty} \frac{n}{1+n^2} = \infty.$$

Since by definition $EZ = EZ^+ - EZ^-$, we conclude that EZ does not exist. To force that EZ exist, we can assume that $E|U| < \infty$. In this case, the RV Z is symmetric, implying that $EZ = 0$.

Problem 2. A Galton-Watson process starts with $X_0 = 1$ and has the offspring distribution $p(0) = 0.1, p(1) = 0.7, p(2) = 0.2$.

(a) Find the probability of extinction.

(b) Find the expected size of the nth generation, $n \in \mathbb{N}$.

(c) Find the variance of the size of the nth generation.

(d) Now assume that $X_0 = m$, where $m \in \mathbb{N}$ is a positive integer, and answer questions (a)-(c) for this case.

Solution: Compute the generating function of the distribution:

$$g(x) = 0.1(1 + 7x + 2x^2).$$

Solving for x the equation $g(x) = x$, we find $P_e = 1/2$.

(b) The expectation of the distribution p is $\mu = 1.1$, and thus the expected size of the nth generation is μ^n.

(c) Since $g_n(x) = g_{n-1}(g(x))$, we find

$$g_n'(x) = g_{n-1}'(g_(x))g'(x)g_n''(x) = g_{n-1}''(g(x))g'(x)^2 + g_{n-1}(g(x))g''(x). \tag{1}$$

Taking $x = 1$ and using the equality $g(1) = 1$, we find

$$g_n''(1) = g_{n-1}''(1)g'(1)^2 + g_{n-1}'(1)g''(1).$$

Since $g'(1) = \mu, g_{n-1}'(1) = \mu^{n-1}$, we obtain $g''(1) = 0.4$ and also

$$g_n''(1) = g_{n-1}''(1)\mu^2 + \mu^{n-1}g''(1) = \cdots = g''(1)\sum_{k=n-1}^{2n-2} \mu^k = g''(1)\mu^{n-1}\frac{\mu^n - 1}{0.1}.$$

Finally,

$$\begin{aligned}\text{Var}(X_n) &= g_n''(1) + EX_n - (EX_n)^2 = g''(1)\mu^{n-1}\frac{\mu^n - 1}{0.1} \\ &= 4(0.1)^{n-1}(0.1^n - 1) - \mu^n(\mu^n - 1) = (4 - \mu)\mu^{n-1}(\mu^n - 1) \\ &= 2.9 \cdot \mu^{n-1}(\mu^n - 1).\end{aligned}$$

(d) The process comprises of m independent GW trees, and it becomes extinct if and only if each of the m branches becomes extinct, so $P_e = (1/2)^m$. Similarly, $EX_n = m \cdot \mu^n$, and $\text{Var}(X_n)$ is m times the variance found in part c.

Problem 3. Consider a Markov chain with state space $\{0, 1, 2, 3 \dots\}$ and transitions

$$\begin{aligned}p_{i,i-1} &= 1, \quad i = 1, 2, 3, \dots \\ p_{0,i} &= p_i, \quad i = 0, 1, 2, 3, \dots\end{aligned}$$

where $p_i > 0$ for all i and $\sum_{i \ge 0} p_i = 1$.

(a) Is this chain irreducible? What is the period of state 0? What is the period of state $i \ge 1$?

(b) What is the condition on the pmf (p_i) that guarantees that the chain is positive recurrent?

(c) Assuming that the condition in (b) is satisfied, what is the expected time of return to state i if the process starts in state i?

Solution: (a) The chain is irreducible aperiodic (the GCD of return times for every state is 1).

(b) Let (π_i) be the stationary pmf. From $\pi P = \pi$ we find

$$\pi_i = \pi_{i+1} + \pi_0 p_i, \quad i \ge 0$$

so

$$\pi_i = \pi_0(1 - p_0 - \dots - p_{i-1}) \quad i \ge 1.$$

Now from

$$\sum_{i=0}^{\infty} \pi_i = \pi_0 \sum_{i=0}^{\infty} (1 - p_0 - \dots - p_{i-1}) = 1$$

we obtain that the stationary pmf exists if and only if this series converges, i.e.,

$$\sum_{i=0}^{\infty} (1 - p_0 - \dots - p_{i-1}) < \infty$$

(equivalently, $E_0 X_1 < \infty$). This is the necessary and sufficient condition for positive recurrence.

(c) If the chain is positive recurrent, the expected time of return to i equals $\frac{1}{\pi_i} = \frac{\sum_{i=0}^{\infty}(1-p_0-\dots-p_{i-1})}{1-p_0-\dots-p_{i-1}}$.

Problem 4. Let X be a homogeneous continuous-time Markov chain with the state space $\{0, 1, 2\}$ and the generator matrix

$$Q = \begin{pmatrix} -3 & 3 & 0 \\ 0 & 0 & 0 \\ 1 & 1 & -2 \end{pmatrix}$$

Find the matrix of transitions $P(t), t \ge 0$ (as a function of t) and the distribution of X_t for all $t > 0$. Assume that the initial distribution of the chain X is uniform.

Solution:

$$P(t) = \begin{pmatrix} e^{-3t} & 1 - e^{-3t} & 0 \\ 0 & 1 & 0 \\ e^{-2t} - e^{-3t} & 1 + e^{-3t} - 2e^{-2t} & e^{-2t} \end{pmatrix}$$

$$p_X(t) = \frac{1}{3}(e^{-2t}, 3 - 2e^{-2t}, e^{-2t})$$

Problem 5. Consider a simple random walk $S_n = S_0 + X_1 + \dots + X_n, n \ge 1$ with $P(X_i = 1) = 1 - P(X_i = -1) = p$ for all i.

In both parts of the problem please compute the answers from the first principles rather than citing formulas from the lectures.

(a) What is the probability that state 2 is reached before state -3, starting from state $S_0 = i \in \mathbb{Z}$, i.e.

$$P(\cup_{n>0}(S_n = 2 | S_0 = i, S_1 \neq -3, \ldots, S_{n-1} \neq -3))$$

for each $i \in \mathbb{Z}$. Compute the numerical value of this probability if $S_0 = 0$ (the walk starts at 0), assuming that $p = 0.7$. (Hint: Denote by r_i the probability of reaching 2 before -3 starting from i, and write equations for $r_i, -3 \leq i \leq 2$).

(b) Find the expected number of steps until the walk reaches the state 2 or -3 for the first time, starting at state $i = \{-3, -2, -1, 0, 1, 2\}$. Please give an answer for each i in this range. Compute the numerical value of this expectation if $S_0 = 0$ (the walk starts at 0), assuming that $p = 0.7$.

Solution:

(a) If $i \geq 2$, the answer is 1 and if $i \leq 3$, it is zero. Otherwise, let $r_i, -3 < i < 2$ be the probability of the event in question. We have (with $p = 0.7, \bar{p} = 0.3$)

$$r_i = p r_{i+1} + \bar{p} r_{i-1}, \quad i = -2, -1, 0, 1,$$

$r_{-3} = 0, r_2 = 1$. Solving this system, we obtain

$$r_i = \frac{(\bar{p}/p)^{i+3} - 1}{(\bar{p}/p)^5 - 1}, \quad -3 \leq i \leq 2,$$

and $r_0 = ((3/7)^3 - 1)/((3/7)^5 - 1) \approx 0.93$.

(b) As in (a), let t_i be the expected time to reach 2 or -3, starting from i. Then

$$t_i = 1 + p t_{i+1} + \bar{p} t_{i-1}, \quad -3 < i < 2.$$

and $t_2 = t_{-3} = 0$. Solving, we obtain

$$t_i = \frac{5}{p - \bar{p}} \frac{(\bar{p}/p)^{i+3} - 1}{(\bar{p}/p)^5 - 1} - \frac{i+3}{p - \bar{p}}, \quad -3 \leq i \leq 2.$$

In particular, $t_0 \approx 4.18$.