ENEE620. Midterm examination, 10/15/2024. Instructor: A. Barg

• Please submit your work to ELMS Assignments as a single PDF file. You must submit your paper within 3 hours from accessing the exam paper online. • Each problem is 10 points. Max score=50 points • Your answers should be justified. Giving just the answer may result in no credit for the problem. • Please pay attention to the writing. You may lose points if your paper is difficult to read.

Problem 1. (Independence of RVs)

(a) Consider the standard probability space $(\Omega = [0, 1], \mathcal{B}, dx)$. In each of the subproblems (a1), (a2), (a3) below we define two random variables X and Y. For each of these three pairs, determine whether the random variables X and Y are independent.

(a1) $X = \omega^2, Y = 1 - \omega^2$;

(a2) $X = 1/2, Y = \omega$;

(a3) $X = \omega, Y(\omega) = \frac{1}{2}\mathbb{1}_{[0,1/4)}(\omega) + \mathbb{1}_{\{1/4\}}(\omega) + \frac{1}{2}\mathbb{1}_{(1/4,3/4)}(\omega) + \frac{1}{4}\mathbb{1}_{\{3/4\}}(\omega) + \frac{1}{2}\mathbb{1}_{(3/4,1]}(\omega)$.

Here $\mathbb{1}_{\{\cdot\}}$ is the indicator of the set in the subscript (an interval or a single point).

(b) Consider the standard probability space $(\Omega = [0, 1], \mathcal{B}, dx)$ and a random variable $X = \omega$. Is it possible to define a *nonconstant* random variable Y on the same space such that X and Y are independent? Hint: Think in terms of the σ-algebras generated by X and Y.

SOLUTION:

(a1) No: $P(X \le 1/4, Y \ge 3/4) \ne P(X \le 1/4)P(Y \ge 3/4)$

(a2) Yes because X does not change with ω.

(a3) We can ignore the values at 1/4 and 3/4 because the probability of those outcomes is 0. We conclude that $P(Y = \frac{1}{2}) = 1$, so this question reduces to (b), with the same answer: the RVs are independent.

(b) We assume that X is nonconstant with probability one. Then the answer is No: the σ-algebra $\sigma(X) = \mathcal{B}$, so it contains any other σ-algebra on this space. Thus for any nonconstant RV Y its σ-algebra $\sigma(Y)$ will depend on $\sigma(X)$.

Problem 2.

(a) Consider a sequence of RVs $(X_n)_n$ that satisfies $\sum_{n\ge1} E|X_n| < \infty$. Show that then $X_n \overset{\text{a.s.}}{\to} 0$.

(b) For a sequence of RVs $(X_n)_n$, suppose that $\sum_{n\ge1} P(X_n > \epsilon) < \infty$. Then show that $\limsup_n X_n \le \epsilon$ a.e.

SOLUTION:

(a) Consider the random variable $X = \sum_{n=1}^\infty |X_n|$. Then $X > 0$ and $EX = E\sum_{n=1}^\infty |X_n| = \sum_{n=1}^\infty E|X_n| < \infty$, where in the last step we used Fubini's theorem. Thus, X is finite, i.e., $X < \infty$ a.s., i.e., the series $\sum_{n=1}^\infty |X_n|$ converges a.s., implying that $|X_n| \to 0$ a.s. But this is possible only if $X_n \to 0$.

(b) Borel-Cantelli implies that $P(X_n > \epsilon \text{ i.o.}) = 0$. In other words, for all but finitely many instances n, $X_n < \epsilon$ a.s. This means that $\inf\limits_{n\ge1} \sup\limits_{m\ge n} X_m < \epsilon$ a.s., as required.

Problem 3.

We are given a sequence of independent random variables $(X_n)_n$ with $P(X_n = 0) = 2/3, P(X_n = 3) = 1/3$.

(a) What is the probability that $\sum_{n=1}^N X_n < M$ i.o. with respect to $N = 1, 2, \dots,$ where $M > 0$ is some fixed positive integer?

(b) What is the distribution of the random variable $Y = \sum_{n=1}^\infty X_n$?

(c) What is the distribution of the random variable $Z = \prod_{n=1}^\infty X_n$?

SOLUTION:

(a) Zero. Indeed, fix $N, M = 3m$, then

$$p_N := P\Big(\sum_{n=1}^{N} X_n \le M\Big) = \sum_{n=1}^{m} \binom{N}{n} (1/3)^n (2/3)^{N-n}.$$

For $N \ge m$, we have

$$p_N \le mN^m (2/3)^{N-m} = (3/2)^m (2/3)^{N-m\log_{3/2} N}$$

This implies that the sum $\sum_{N=1}^{\infty} p_N < \infty$, so the answer is indeed zero by the Borel-Cantelli lemma.

(b) Thus for any fixed M, Y is greater than M with probability 1. So $Y = \infty$ a.s.

Alternatively, $p_N \to 0$ as $N \to \infty$, and this is true for any fixed M. Thus

$$P(Y = \infty) = P\Big(\lim_{N\to\infty} \sum_{n=1}^{N} X_n \ge M\Big) = 1,$$

so $Y = \infty$ a.s.

(c) On the other hand, $P(Z > 0) = \lim_{n\to\infty} P(X_1 = 3, X_2 = 3, \ldots, X_n = 3) = \lim_n (1/3)^n = 0$, so $Z = 0$ a.s.

Problem 4. The random variable X is supported on the interval (a, b), i.e., $P(a < X < b) = 1$, where $0 < a < b < \infty$ are some real numbers. Find

(a) $\sup_X E[(X - EX)^2]$,

(b) $\sup_X (EX)\cdot E(1/X)$,

(c) Is it possible that $EX\cdot E(1/X) = 1/2$ for some random variable X as described above ?

where in (a), (b) the supremum is taken over all possible RVs X that satisfy the stated assumption (are supported on (a, b)).

Hint: One way to solve parts (a), (b) of problem is to use convexity. For a function $f(x)$ convex on the segment (a, b), the value $f(x), a < x < b$ lies below the straight line segment that connects $f(a)$ and $f(b)$, i.e.,

$$f(x) \le \frac{x-a}{b-a} f(b) + \frac{b-x}{b-a} f(a).$$

Use this for $f(x) = (x - EX)^2$ in (a) and for $f(x) = 1/x$ in (b).

Solution: (a) Taking the hint, $f(x) = (x - EX)^2$ is convex, so

$$(X - EX)^2 \le \frac{X-a}{b-a}(b - EX)^2 + \frac{b-X}{b-a}(a - EX)^2.$$

Taking the expectation on both sides,

$$E[(X - EX)^2] \le \sup_c \Big\{\frac{c-a}{b-a}(b-c)^2 + \frac{b-c}{b-a}(a-c)^2\Big\} = \sup_c (c-a)(b-c) = \frac{(b-a)^2}{4}.$$

We can take $P(X = a) = P(X = b) = 1/2$, then $EX^2 - (EX)^2 = \frac{a^2+b^2}{2} - \frac{(a+b)^2}{4} = \frac{(b-a)^2}{4}$, so this X yields the maximum variance.

(b) Again using the hint for $f(x) = 1/x$:

$$\frac{1}{X} \le \frac{X-a}{b-a}\frac{1}{b} + \frac{b-X}{b-a}\frac{1}{a}$$

$$EX\cdot E(1/X) \le \sup_c \Big\{\frac{c-a}{b-a}\frac{c}{b} + \frac{b-c}{b-a}\frac{c}{a}\Big\} = \frac{(a+b)^2}{4ab}$$

The same random variable as in (a) yields $EX = (a + b)/2$, $E(1/X) = \frac{1}{2a} + \frac{1}{2b} = \frac{a+b}{2ab}$, attaining the maximum.

(c) No, since the product satisfies the inequality $EX\cdot E(1/X) \ge 1$. Indeed, by convexity of $f(x) = 1/x$ we have

$$\frac{1}{EX} \le E\frac{1}{X}.$$

Problem 5. (LLN)

Below $(X_n)_n$ is a sequence of independent random variables.

(a) If $P(X_n = 2^n) = P(X_n = 2^{-n}) = 1/2, n \geq 1$, does this sequence satisfy the (strong or weak) law of large numbers?

(b) If $P(X_n = -\sqrt{n}) = P(X_n = \sqrt{n}) = \frac{1}{2n}$ and $P(X_n = 0) = 1 - \frac{1}{n}, n \geq 1$, does this sequence satisfy the (strong or weak) law of large numbers?

SOLUTION:

(a). The RVs are not identically distributed, and the expectation $EX_n \approx 2^{n-1} \to \infty$ as $n \to \infty$. We guess that $S_n := \sum_{k=1}^{n} X_k$ satisfies $S_n/n \to \infty$ a.s. As in Problem 3, we argue that $P(S_n/n > M) \to 1$ for any fixed real number M; thus, $P(\lim_{n\to\infty} \frac{S_n}{n} = \infty) = 1$.

This sequence does not satisfy the LLNs as discussed in class.

(b) $EX_n = 0$ and $EX_n^2 = 1$, so using Chebyshev's inequality,

$$P(|X_n| \geq \epsilon) \leq \frac{1}{n\epsilon^2} \to 0;$$

the sequence satisfies the WLLN. In class we stated the theorems only for identically distributed RVs, which the X_ns are not.

Next, let us consider SLLN. We cannot use Kolmogorov's theorem because again the RVs are not identically distributed. Let us show directly that $P(\lim_{n\to\infty} \frac{S_n}{n} = 0) = 1$. Since $EX_n = 0, EX_n^2 = 1, EX_n^4 = n$,

$$ES_n^4 = \sum_{m=1}^{n} EX_n^4 + \sum_{i<j} \binom{4}{2} EX_i^2 EX_j^2 = n^2 + 3n(n-1)$$

and

$$P\Big(\frac{S_n}{n} \geq \epsilon\Big) \leq \frac{ES_n^4}{(\epsilon n)^4} \leq \frac{4}{\epsilon^2 n^2}.$$

This implies that $\sum_{n\geq 1} P(\frac{S_n}{n} \geq \epsilon) < \infty$ or $P(\frac{S_n}{n} \geq \epsilon \text{ i.o.}) = 0$, and by symmetry, the same is true for $\leq -\epsilon$. Thus, $P(\limsup_n \frac{|S_n|}{n} = 0) = 1$, implying that the sequence satisfies SLLN.