ENEE620. Midterm examination, 10/12/2023. Instructor: A. Barg

- Please submit your work to ELMS Assignments as a single PDF file by Oct.12, **6:00pm** EDT.
- Each problem is 10 points. Max score=50 points
- Your answers should be justified. Giving just the answer may result in no credit for the problem.
- Please pay attention to the writing. You may lose points if your paper is difficult to read.

Problem 1.

(a) We say that a sequence of events $(A_n)_n$ converges if

$$\limsup_{n\to\infty} A_n = \liminf_{n\to\infty} A_n,$$

and we denote the event defined by this equation as $\lim_n A_n$.

Suppose that a sequence of events $(A_n)_n$ converges. Let $\mathcal{A}_m := \cap_{n\ge m} A_n, \mathcal{B}_m := \cup_{n\ge m} A_n$. Use monotonicity of the sequences $\mathcal{A}_m$ and $\mathcal{B}_m$ to show that

$$\lim_{n\to\infty} P(A_n) = P(\lim_{n\to\infty} A_n).$$

(b) Let (A_n) be a sequence of events, and let $E, F \in \mathcal{F}$ be two events such $E \subseteq A_n \subseteq F$ for all $n \ge 1$. We are given that $P(E) = P(F)$. Using these assumptions, show that

$$P(E) = P(\liminf_{n\to\infty} A_n) = P(\limsup_{n\to\infty} A_n) = P(F).$$

SOLUTION:

(a) Taking the hint, observe that $\mathcal{A}_m \subset \mathcal{A}_{m+1}$ and $\mathcal{B}_m \supset \mathcal{B}_{m+1}$. Then by continuity of probability, $\lim_{m\to\infty} P(\mathcal{A}_m) = P(\cup_{m\ge1}\mathcal{A}_m) = P(\liminf_n A_n)$ and likewise, $\lim_m P(\mathcal{B}_m) = P(\limsup_n A_n)$. By the convergence assumption we have $P(\liminf_n A_n) = P(\limsup_n A_n)$, so

$$\lim_n P(\mathcal{A}_n) = \lim_n P(\mathcal{B}_n).$$

Since $P(\mathcal{A}_n) \le P(A_n) \le P(\mathcal{B}_n)$, taking the limit of $n \to \infty$ and using the previous line, we obtain that

$$\lim_n P(A_n) = \lim_n P(\mathcal{A}_n) = \lim_n P(\mathcal{B}_n) = P(\lim_n A_n).$$

(b) Since $E \subseteq A_n$, also $E \subseteq \cup_{n\ge m} A_n$ for all $m \ge 1$, and then $E \subseteq \liminf_n A_n$. Similarly, $\limsup_n A_n \subset F$, and then

$$P(E) \le P(\liminf_n A_n) \le P(\limsup_n A_n) \le P(F).$$

Together with the condition $P(E) = P(F)$, this completes the proof.

Problem 2.

(a) Given an RV X on a probability space $(\Omega, \mathcal{F}, P)$, we say that the sigma-algebra $\mathcal{A}$ is *generated* by X if it is the smallest sigma-algebra that contains all events of the form $X^{-1}(B) \in \mathcal{F}$ for all Borel sets $B \in \mathcal{B}(\mathbb{R})$. Note that generally $\mathcal{A}$ is not the same as (is smaller than) $\mathcal{F}$.

We consider the standard probability space $(\Omega = [0,1], \mathcal{F} = \mathcal{B}([0,1]), dx)$.

(1) Give a complete description of the sigma-algebra generated by the RV $X_1 = \omega/4$.

(2) Give a complete description of the sigma-algebra generated by the RV $X_2 = 2/3$.

(3) Give a complete description of the sigma-algebra generated by the RV

$$X_3 = \begin{cases} \frac{1}{3}, & \omega \in [0,1/3), \\ 1/2, & \omega \in [1/3,2/3), \\ 1 & \omega \in [2/3,1]. \end{cases}$$

(b) Give the expressions for the distribution functions $F_{X_1}(x), F_{X_2}(x), F_{X_3}(x)$ of the random variables defined in Part (a). Make sure to give answers for all $x, -\infty < x < \infty$.

SOLUTION:

(a) (1) Given an interval $[a, b], 0 \le a \le b \le 1/4$, take $B = [a, b]$, then $X_1^{-1}(B) = [4a, 4b] \in \mathcal{F}$. The smallest sigma-algebra on Ω generated by all closed intervals is $\mathcal{B}([0, 1])$, and thus $\mathcal{A} = \mathcal{F}$.

(2) Using the definition, either $(2/3) \in B$, and then $X_2^{-1}(B) = [0, 1]$, or $(2/3) \notin B$ and then $X^{-1}(B) = \emptyset$. Thus $\mathcal{A} = \{\Omega, \emptyset\}$.

(3) For any $x > 1$, $X_3^{-1}(-\infty, x] = \Omega$, and for any $x < 1/3$, $X_3^{-1}(-\infty, x] = \emptyset$. For $x \in [0, 1/3)$, $X_3^{-1}(-\infty, x] = [0, 1/3)$, so $[0, 1/3) \in \mathcal{A}$, and similarly $[1/3, 2/3) \in \mathcal{A}$ and $[2/3, 1] \in \mathcal{A}$. Taking their unions and complements, we obtain the answer, given below.

Answer: $\mathcal{A} = \{\Omega, \emptyset, [0, 1/3), [1/3, 1], [1/3, 2/3), [1/3, 2/3)^c, [2/3, 1], [0, 2/3)\}$.

(b) $F_{X_1}(x) = 0, x \le 0$; $F_{X_1}(x) = 4x, 0 \le x \le 1/4$; $F_{X_1}(x) = 1, x \ge 1/4$;

$F_{X_2}(x) = 0, x < 2/3$; $F_{X_2}(x) = 1, x \ge 2/3$ (this CDF corresponds to the $\delta_{2/3}$ measure);

$F_{X_3}(x) = 0, x < 1/3$; $F_{X_3}(x) = 1/3, 1/3 \le x < 1/2$; $F_{X_3}(x) = 2/3, 1/2 \le x < 1$; $F_{X_3}(x) = 1, x \ge 1$.

Problem 3.

For parts (a)-(c), assume that $F(x)$ is a continuous distribution function.

(a) Show that $\int_{-\infty}^{\infty} F(x)dF(x) = \frac{1}{2}$ (a change of variable should work),

(b) Similarly, show that $\int_{-\infty}^{\infty} F^k(x)dF^n(x) = \frac{n}{n+k}$. Here $F^k(x) = (F(x))^k$ is the kth power of $F(x)$, same for F^n.

(c) Now let X be an RV for which $F(x)$ is the CDF. What is the distribution of the random variable $F(X)$?

(d) Let X and Y be independent RVs on $(\Omega, \mathcal{F}, P)$ with CDFs $F_X(x)$ and $F_Y(x)$ such that $P(Y = 0) = 0$. What is the CDF of the RV $Z = X/Y$.

Notes for Part (d): (i) Both F_X and F_Y may not have densities, so do no assume that they have them. (ii) Recall that for an RV U we have $P(a \le U \le b) = \int_a^b dF_U(x)$.

SOLUTION:

(a) Let $y = F(x)$, then $y \in [0, 1]$, and $\int_0^1 y dy = \frac{1}{2}$.

(b) As above, $\int_0^1 y^k d(y^n) = \int_0^1 y^k n y^{n-1} dy = n \int_0^1 y^{n+k-1} dy = \frac{n}{n+k}$.

(c) Let $Z = F(X)$, then $P(Z \le z) = P(F(X) \le z) = P(X \le F^{-1}(z)) = F(F^{-1}(z)) = z$. This implies that $Z \sim$ Unif$[0, 1]$. Note that F does not have to be strictly monotone increasing. If $F(x) = c$ on some interval of the real line, then we can take an arbitrary x such that $F(x) = c$ as the value of $F^{-1}(c)$, e.g., $x = \inf\{x' : F(x') \ge c\}$, and the proof still goes through.

(d) We have $X/Y \le z$, translating into $X \le zY$ if $Y > 0$ and $X \ge zY$ if $Y < 0$, and so

$$\begin{aligned} P(Z \le z) = P(X/Y \le z) &= P(X \le zY, Y > 0) + P(X \ge zY, Y < 0) \\ &= \int_0^{\infty} F_X(zy) dF_Y(y) + \int_{-\infty}^{0} (1 - F_X(zy)) dF_Y(x). \end{aligned}$$

Problem 4.

Let $X_i, 1 \le i \le n$ be independent RVs distributed uniformly on $[0, 1]$, and let $U_n = \min_{1 \le i \le n} X_i, V_n = \max_{1 \le i \le n} X_i$.

(a) Find the distribution of U_n. Does the sequence $(U_n)_n$ converge as $n \to \infty$? If yes, what is the limit, and what is the type of convergence?

(b) Find the CDF of V_n. Show that as $n \to \infty$, the sequence $n(1 - V_n)$ converges in law to the exponential distribution Exp(1).

SOLUTION:

(a) z $P(U_n \le \epsilon) = 1 - P(U_n > \epsilon) = 1 - (1-\epsilon)^n$, and thus $\lim_{n\to\infty} P(U_n \le \epsilon) = 1$ for any $\epsilon > 0$. In other words, $U_n \xrightarrow{p} 0$.

Moreover, $P(U_n \ge \epsilon) = (1-\epsilon)^n$, and $\sum_{n\ge 1}(1-\epsilon)^n < \infty$, so $P(U_n \ge \epsilon \text{ i.o.}) = 0$; thus with probability one, $\lim_{n\to\infty} U_n = 0$. This means that $U_n \xrightarrow{\text{a.s.}} 0$.

Also, $P(U_n^2 \ge \epsilon) = P(U_n > \sqrt{\epsilon}) = (1-\sqrt{\epsilon})^n$ for $0 \le \epsilon \le 1$, so

$$EU_n^2 = \int_0^\infty P(U_n^2 > z)dz = \int_0^1 (1-\sqrt{z})^n dz = \frac{2}{(n+1)(n+2)} \to 0$$

as $n \to \infty$, so $U_n \to 0$ m.s. (and also in L_p).

(b) Since $V_n \le z$ if and only if all $X_i \le z$, we have $P(V_n \le z) = (F_X(z))^n = z^n$. Now, as $n \to \infty$,

$$P(n(1-V_n) \le z) = P(V_n \ge (1-\frac{z}{n})) = 1 - \left(1-\frac{z}{n}\right)^n \to 1 - e^{-z}.$$

This is the CDF of the exponential law with $\lambda = 1$, so $n(1-V_n) \xrightarrow{d} \text{Exp}(1)$.

Problem 5.

For an RV Y we denote $\sigma(R) := \sqrt{\text{Var}(Y)}$.

(a) Assuming that the expectations and variances exist, show that

$$1 \le EX \cdot E\Big(\frac{1}{X}\Big) \le 1 + \sigma(X)\sigma(1/X).$$

When does equality $EX \cdot E(\frac{1}{X}) = 1$ hold?

Note: For the left-hand side, use convexity and Jensen's inequality; for the right-hand side try Cauchy-Schwarz.

(b) X is a nonnegative, integer-valued RV, i.e., $X(\omega) \in \{0, 1, 2, \dots\}$. Show that $P(X > 0) \ge \frac{(EX)^2}{EX^2}$. For this, notice that $EX = E(\mathbb{1}_{\{X\ne 0\}}X)$ and use Cauchy-Schwarz.

SOLUTION:

(a) We will assume that $EX \ne 0$. For a complete proof, we must assume that $X > 0$ a.s. or $X < 0$ a.s. In the first case, also $EX > 0$, and we need to show that $\frac{1}{EX} \le E\frac{1}{X}$. Observe that $(\frac{1}{y})'' = \frac{2}{y^3} > 0$ for $y > 0$, so the function $g(y) = \frac{1}{y}$ is strictly $\cup$-convex. Our RV is integrable by the problem statement, so Jensen's inequality applies. It says that $g(E(X)) \le Eg(X)$; in our case exactly that

$$\frac{1}{EX} \le E\frac{1}{X}.$$

Now $X < 0$ and $EX < 0$, then we need to show that $\frac{1}{EX} \ge \frac{1}{X}$, because this again implies that $EX\frac{1}{EX} \ge 1$. This works because the function $y \mapsto \frac{1}{y}$ is strictly concave for $y < 0$, so using Jensen's, we have $g(E(X)) \ge Eg(X)$. This finishes the proof.

If $X > 0$ a.s. or $X < 0$ a.s., then $EX \cdot EX^{-1} = 1$ holds if and only X is a constant (then $\sigma(X) = \sigma(1/X) = 0$). If X is allowed to take both positive and negative values, then this conclusion is wrong: for instance, take X with $P(X = -1) = 1/9$, $P(X = \frac{1}{2}) = P(X = 2) = 4/9$.

Next, by Cauchy-Schwarz, $E((X - EX)(Y - EY))^2 \le E(X - EX)^2 E(Y - EY)^2$, which shows that

$$\frac{E((X - EX)(Y - EY))}{\sigma(X)\sigma(Y)} \ge -1.$$

Thus,

$$-\sigma(X)\sigma(Y) \le E((X - EX)(Y - EY)) = E(XY) - EXEY.$$

Take $Y = 1/X$, then $E(XY) = 1$, and

$$EX \cdot E\frac{1}{X} \le 1 + \sigma(X)\sigma(1/X).$$

(b) Taking the hint, $(EX)^2 = E(\mathbb{1}_{X\ne 0}X)^2 \le E(\mathbb{1}_{X\ne 0}^2)EX^2 = P(X \ne 0)EX^2$.