ENEE620. Midterm examination, 10/15/2020. Instructor: A. Barg

- Please submit your work as a single PDF file to ELMS/Canvas Assignments by 10/15, 4pm US Eastern time.
- Each problem is 10 points. Max score=50 points
- Your answers should be justified. Giving just the answer may result in no credit for the problem.
- Please pay attention to the writing. You may lose points if your paper is difficult to read.

Problem 1. Let X, Y be real-valued RVs on a probability space $(\Omega, \mathcal{F}, P)$, which may or may not be dependent. Show that for any Borel set $B \in \mathcal{B}(\mathbb{R})$

$$|P(X \in B) - P(Y \in B)| \le P(X \ne Y).$$

(this is a basic Venn diagram-type argument).

Problem 2. (a) Let $Y \sim$ Unif$[0,1]$ be an RV uniformly distributed on the unit segment $[0,1]$. Define the RVs $X_n = \sqrt{n}\mathbb{1}_{\{Y \le 1/n\}}, n = 1, 2, \dots,$ where $\mathbb{1}_{\{\cdot\}}$ is the indicator of the event in the subscript.

Does the sequence $(X_n)_{n\ge1}$ converge in probability? almost surely? in the mean-square sense?

(b) Now let $Y_n \sim$ Unif$[0,1], n = 1, 2, \dots,$ be i.i.d. RVs and let $X_n = \sqrt{n}\mathbb{1}_{\{Y_n \le 1/n\}}, n = 1, 2, \dots.$ Does the sequence $(X_n)_{n\ge1}$ converge in probability? almost surely? in the mean-square sense?

Problem 3. Consider a sequence of i.i.d. RVs $X_1, X_2, \dots$ with CDF F. Let $\lambda_1, \lambda_2, \dots$ be a sequence of numbers with $\lambda_n \to \infty$. Consider the sequence of events $A_n = \{\max_{1\le m\le n} X_m > \lambda_n\}, n = 1, 2, \dots.$ Your task is to establish whether the events A_n occur infinitely often. Namely, show that

$$P(A_n \text{ i.o.}) = \begin{cases} 0 & \text{if } \sum_{n\ge1}(1 - F(\lambda_n)) < \infty \\ 1 & \text{if } \sum_{n\ge1}(1 - F(\lambda_n)) = \infty. \end{cases}$$

Problem 4.
(a) Let X be Geometric(p). Show that

$$E\Big(\frac{1}{1+X}\Big) = \ln((1-p)^{\frac{p}{p-1}}).$$

(b) For $\lambda > 0$ let X_n be Geometric$(\frac{\lambda}{\lambda+n}), n = 1, 2, \dots.$ Show that X_n/n converges in distribution to an exponential distribution and find the parameter of the limit distribution.

Problem 5. Let X, Y be independent RVs in $\mathbb{N}$ with $p_X(n) = p_Y(n) = 2^{-n}, n = 1, 2, \dots.$ Find

$$(a) P(\min(X,Y) \le n);$$
$$(b) P(X = Y);$$
$$(c) P(X > Y);$$
$$(d) P(X \text{ divides } Y);$$
$$(e) P(X \ge kY), \text{ where } k \text{ is a given positive integer.}$$

The answers to parts (a),(b),(d),(e) should not include any infinite sums.