ENEE620. Final exam, 12/16/2024. Instructor: A. Barg

- Please submit your work to ELMS Assignments as a single PDF file. You must submit your paper within 3 hours from accessing the exam paper online.
- Each problem is 10 points. Max score=50 points
- Your answers should be justified. Giving just the answer may result in no credit for the problem.
- Please pay attention to the writing. You may lose points if your paper is difficult to read.

Problem 1. Consider a Markov chain $(X_n)_n$ on a countable state space S. Let $s \in S$ and

$$t_x^k(y) = \inf\{n > t_x^{k-1}(y) : X_n = y\}, k \ge 1,$$

where $t_x^0(y) := 0$ and where the subscript x means that the process starts from state $x \in S$. Let $p_{xy} := P(t_x^1(y) < \infty)$.

(a) Show that for each $k \ge 1$, $P(t_x^k(y) < \infty) = p_{xy} p_{yy}^{k-1}$. Please give a formal proof. Did you need to assume that the chain is recurrent? Justify your answer.

(b) Now show that the number $N_y := |\{n : X_n = y, n \ge 1\}|$ of visits to y satisfies $E_x(N_y) = \frac{p_{xy}}{1-p_{yy}}$.

SOLUTION: (a) If $p_{xy} = 0$, then all the probabilities $P(t_x^k(y) < \infty)$ are zero, proving the statement. If $p_{yy} = 0$, then $P(t_x^k(y) < \infty) = 0$ except for $k = 1$, again proving the statement. Now suppose that neither p_{xy} nor p_{yy} are zero. By the problem statement, $P(t_x^1(y) < \infty) = p_{xy}$. We use this as induction base.

Now let $k \ge 1$ and suppose that we have proved $P(t_x^{k-1}(y) < \infty) = p_{xy} p_{yy}^{k-2}$. We use the strong Markov property to perform the induction step. Since we now have that $p_{xy}, p_{yy} > 0$, $P(t_x^k < \infty) > 0$ for all $k \ge 1$. Define $\mathcal{F}_n = \sigma(X_0, X_1, \dots, X_n), n \ge 1$. The *strong Markov property* tells us that for all $k \ge 1$

$$P(X_{t_x^k+1} = z | \mathcal{F}_{t_x^k}) = P_{X_{t_x^k}}(X_{t_x^k+1} = z) = P(X_{t_x^k}, z),$$

where $P(X_{t_x^k}, z)$ is the transition probability from the (random) state $X_{t_x^k}$ to z. In our case, $X_{t_x^{k-1}} = y$, and the probability $P(X_{t_x^{k-1}}, y) = p_{yy}$,

$$P(t_x^k(y) < \infty) = P(t_x^{k-1}(y) < \infty) p_{yy} = p_{xy} p_{yy}^{k-1}.$$

Since our argument does not depend on $p_{yy} = 1$, it covers both the recurrent and transient cases, so we do not need the recurrence assumption.

(b)

$$EN_y = E\Big(\sum_{n>0} \mathbb{1}(X_n = y)\Big) = E\Big(\sum_{k\ge1} \mathbb{1}(t_x^k(y) < \infty)\Big) = \begin{cases} \frac{p_{xy}}{1-p_{yy}} & p_{yy} < 1 \\ \infty & p_{yy} = 1 \end{cases}.$$

Problem 2. Let $(X_k, k \ge 1)$ be a sequence of i.i.d. RVs with a finite mean.

(a) Define the RVs

$$S_n := \frac{1}{n} \sum_{k=1}^n X_k X_{k+1},\ n \ge 1.$$

Is the claim *Kolmogorov's SLLN theorem implies that* $S_n \overset{\text{a.s.}}{\to} (EX_1)^2$ true? Explain your answer.

(b) Define the sequences $Y_k = X_{2(k-1)+1} X_{2k}, k \ge 1$ and $Z_k = X_{2k} X_{2k+1}, k \ge 1$. Do the sequences

$$\frac{1}{m} \sum_{k=1}^m Y_k, \quad \frac{1}{m} \sum_{k=1}^m Z_k, \quad m \ge 1$$

converge; if yes, then how, to which limits, and for what reason?

(c) Use the result of Part (b) to prove that $S_n \overset{\text{a.s.}}{\to} (EX_1)^2$.

SOLUTION:

(a) Since the RVs X_1X_2 and X_2X_3 are not independent, SLLN cannot be used directly. At the same time, the dependence is weak and the statement of interest is still true, as shown below.

(b) Both sequences Y_k and Z_k are formed of iid RV's, and thus the sample averages $\frac{1}{n}\sum_{k=1}^n Y_k$ and $\frac{1}{n}\sum_{k=1}^n Z_k$ converge to $(EX)^2$ a.s.

(c) For $n = 2m$ we have

$$S_n = \frac{1}{2m}\sum_{k=1}^m Y_k + \frac{1}{2m}\sum_{k=1}^m Z_k$$

and for $n = 2m+1$ we have

$$S_n = \frac{m+1}{2m+1}\frac{1}{m+1}\sum_{k=1}^{m+1} Y_k + \frac{m}{2m+1}\frac{1}{m}\sum_{k=1}^m Z_k.$$

In both cases when $m \to \infty$ we obtain that $S_n \overset{\text{a.s.}}{\to} \frac{1}{2}(EX)^2 + \frac{1}{2}(EX)^2 = (EX)^2$.

Problem 3. Let X and Y be independent RVs.
(a) Suppose that a number a satisfies $0 < P(X > a) < 1$. Show that then $\{X > a\} \notin \sigma(Y)$.
(b) Show that if $\{X > Y\} \in \sigma(Y)$ then there exist numbers a, b such that $P(a \le X \le b) = 1$ and either $a = b$ or $P(a < Y < b) = 0$.
(c) Show that if the distribution of Y is continuous and such that $P(Y \in \text{supp}(X)) > 0$, then $\{X < Y\} \notin \sigma(Y)$.
The last page of this document reproduces a ChatGPT "solution" of this problem. Your task is to go over it step by step and either argue that the step is correct (if so, make it formal) or not (in this case, replace with a correct argument).

Problem 4. Let $U \sim \text{Unif}[0,1]$ be a uniform RV. Define a sequence of RVs $X_n, n \ge 0$ where $X_n = k2^{-n}$ for the unique integer k such that $k2^{-n} \le U < (k+1)2^{-n}$. Let $f : [0,1] \to \mathbb{R}$ be a bounded measurable function and put $f(x) = f(1)$ for all $x \ge 1$. Finally, define

$$Y_n = 2^n(f(X_n + 2^{-n}) - f(X_n)), \quad n \ge 0.$$

(a) Find the conditional distribution $P(U|X_0, X_1, \dots, X_n)$.
(b) Prove that the sequence $(Y_n)_n$ forms a martingale with respect to the filtration given by the sequence (X_n).
(c) Does the sequence $(Y_n)_n$ converge a.s., and if yes, what is the limit?

SOLUTION:

(a) The sequence X_n forms a sequence of progressively more accurate approximations of U. Suppose we know $X_0, \dots, X_n$, which means that we also know $k_0, \dots, k_n$, i.e., in particular, we know that $k_n 2^{-n} \le U < (k_n+1)2^{-n}$, and furthermore, $P(U|X_0^n) = P(U|X_n)$. If $k_n < 2^n$, then this means that

$$U|X_0^n \sim \text{Unif}[k_n 2^{-n}, (k_n+1)2^{-n}) = \text{Unif}[X_n, X_n + 2^{-n}).$$

If $k_n = 2^n$, then $P(U = 1|X_n) = 1$.

(b) Check that $E|Y_n| < \infty$. Since $f(x) \le N$ for some $N > 0$, $E|Y_n| \le 2^n(2N) < \infty$ for all n. Suppose that $k_n < 2^n$, so $X_n = k_n 2^{-n}$ and

$$X_{n+1} = \begin{cases} \frac{2k_n}{2^{n+1}} & \text{w.p.1/2} \\ \frac{2k_n+1}{2^{n+1}} & \text{w.p.1/2} \end{cases}$$

Then compute the martingale condition:

$$\begin{aligned} E(Y_{n+1}|\mathcal{F}_n) &= 2^{n+1}\Big[\frac{1}{2}\Big(f\Big(\frac{2k_n+1}{2^{n+1}}+\frac{1}{2^{n+1}}\Big)-f\Big(\frac{2k_n+1}{2^{n+1}}\Big)\Big) \\ &+\frac{1}{2}\Big(f\Big(\frac{2k_n}{2^{n+1}}+\frac{1}{2^{n+1}}\Big)-f\Big(\frac{2k_n}{2^{n+1}}\Big)\Big)\Big] \\ &= 2^n\Big[f\Big(\frac{k_n}{2^n}+\frac{1}{2^n}\Big)-f\Big(\frac{k_n}{2^n}\Big)\Big] \\ &= Y_n. \end{aligned}$$

(c) The sequence (X_n) converges a.s. to U since $P(\lim_{n\to\infty}|X_n-U|=0)=1$. However boundedness of f alone does not guarantee convergence of (Y_n). If f is differentiable at every point of $[0,1]$, then $2^n(f(x+2^{-n})-f(x))\to f'(x)$, and $Y_n \overset{\text{a.s.}}{\to} f'(U)$. Otherwise we cannot claim that Y_n is L_1-bounded since $\sup_n E(Y_n)$ is not controlled by the boundedness of f alone, so the sufficient condition for martingale convergence does not apply.

Problem 5. Let $(S_n, n\ge 0)$ be a sequence of random variables defined as $S_0=0$ and $S_n=S_0+\sum_{i=1}^n X_n$, where the increments X_n are i.i.d.

(a) Suppose that the increments have the distribution $P(X=1)=p=1-P(X=-1), 0<p<1$, i.e., $(S_n)_n$ is a biased random walk. Let λ be a real number. Find the value(s) $\gamma\in\mathbb{R}$ for which the sequence $e^{\gamma S_n-\lambda n}, n\ge 0$ forms a martingale with respect to the natural filtration.

(b) Now let $(X_n)_n$ be i.i.d. $\mathcal{N}(0,1)$ Gaussian RVs. Show that the sequence $(Y_n=e^{uS_n-nu^2/2}, n\ge 0)$, where $u\in\mathbb{R}$, forms a martingale with respect to the natural filtration.

SOLUTION:

(a) Let $\mathcal{F}_n=\sigma(X_1^n), n\ge 1$. We have

$$E(e^{\gamma S_{n+1}-(n+1)\lambda}|\mathcal{F}_n)=E(e^{\gamma S_n+\gamma X_{n+1}-(n+1)\lambda}|\mathcal{F}_n)=e^{\gamma S_n-(n+1)\lambda}E(e^{\gamma X_{n+1}}).$$

Now, note that $E(e^{\gamma X_{n+1}})=pe^\gamma+(1-p)e^{-\gamma}$, and let us choose γ such that $pe^\gamma+(1-p)e^{-\gamma}=e^\lambda$, or

$$\gamma=\ln\Big(e^\lambda\pm\sqrt{e^{2\lambda}-4p(1-p)}\Big)-\ln(2p)$$

With each of these choices,

$$E(e^{\gamma S_{n+1}-(n+1)\lambda}|\mathcal{F}_n)=e^{\gamma S_n-\lambda n},$$

confirming the martingale property. Moreover, since the RVs $e^{\gamma S_n-\lambda n}$ are nonnegative, the martingale property implies integrability:

$$E|e^{\gamma S_n-\lambda n}|=E[E[e^{\gamma S_n-\lambda n}|\mathcal{F}_n]]=E[e^{\gamma S_{n-1}-\lambda(n-1)}]=\cdots=1.$$

(b) Integrability follows as in part (a) once we establish the martingale property. To prove it, write

$$\begin{aligned} E(e^{uS_{n+1}-(n+1)u^2/2}|\mathcal{F}_n) &= e^{uS_n-(n+1)u^2/2}E(e^{uX_{n+1}}|\mathcal{F}_n) \\ &= e^{uS_n-(n+1)u^2/2}E(e^{uX_{n+1}})=e^{uS_n-(n+1)u^2/2}e^{u^2/2}=Y_n \end{aligned}$$

as required. Here we used the expression for the moment generating function of the Gaussian RV:

$$E(e^{uX_{n+1}})=\frac{1}{\sqrt{2\pi}}\int e^{us-\frac{s^2}{2}}ds=\frac{e^{\frac{u^2}{2}}}{\sqrt{2\pi}}\int e^{-\frac{(s-u)^2}{2}}ds=e^{\frac{u^2}{2}}.$$

AI-provided "solution" of Problem 3:

(a) Suppose $0 < P(X > a) < 1$. Show that $\{X > a\} \notin \sigma(Y)$.

Proof: The event $\{X > a\}$ depends only on the random variable X. Since X and Y are independent, their σ-algebras $\sigma(X)$ and $\sigma(Y)$ are independent as well. If $\{X > a\} \in \sigma(Y)$, then $\{X > a\}$ would be determined by Y. This would violate the independence of X and Y, as the event $\{X > a\}$ has non-trivial probability ($0 < P(X > a) < 1$) and depends solely on X.

Thus, $\{X > a\} \notin \sigma(Y)$.

The ideas are correct, but the presentation is not rigorous enough. The statements "If $\{X > a\} \in \sigma(Y)$, then $\{X > a\}$ would be determined by Y" and "This would violate the independence of X and Y" are not well-formulated mathematical claims. Formally, assume for contradiction that $\{X > a\} \in \sigma(Y)$, so there is a $\sigma(Y)$-measurable event A such that $A = \{X > a\}$ (more precisely, there is a set $B \in \mathcal{B}(\mathbb{R})$ such that $A = \{\omega : X(\omega) \in B\}$). The sigma-algebras $\sigma(X)$ and $\sigma(Y)$ are independent. By definition, the event $\{X > a\} \in \sigma(X)$ and is independent of A. Then

$$P(\{X > a\}) = P(\{X > a\}, A) = P(\{X > a\})P(A) = P(\{X > a\})^2,$$

so $P(\{X > a\}) = 0$ or 1. This contradicts the assumption $0 < P(X > a) < 1$, so $\{X > a\} \notin \sigma(Y)$.

(b) Show that if $\{X > Y\} \in \sigma(Y)$, then there exist numbers a, b such that $P(a \le X \le b) = 1$ and either $a = b$ or $P(a < Y < b) = 0$.

Proof:

(1) Assume $\{X > Y\} \in \sigma(Y)$. This means the event $\{X > Y\}$ is measurable with respect to Y, i.e., there exists a Y-measurable set A such that:

$$\{X > Y\} = \{Y \in A\}.$$

This is correct and well stated.

(2) **Independence of X and Y:** If X and Y are independent, then the probability of the event $\{X > Y\}$ must depend only on the marginal distribution of X, not on Y.

For a fixed value $Y = y$, $P(X > y)$ is fully determined by the distribution of X. The fact that $P(X > Y)$ depends only on P_X does not immediately follow. Anyway, this step can be discarded.

(3) **Implication of $\{X > Y\} \in \sigma(Y)$:**

- For $\{X > Y\}$ to depend only on Y, X must take values in a deterministic set relative to Y. Specifically, there must exist two constants a and b such that $P(a \le X \le b) = 1$. This forces the distribution of X to be degenerate or concentrated in some deterministic range.

(4) **Additional constraint:**

- If $a < b$, then the event $\{X > Y\}$ implies that $P(a < Y < b) = 0$, as otherwise $\{X > Y\}$ would depend on random variability of X outside the control of Y, contradicting $\{X > Y\} \in \sigma(Y)$.

Informal; reasoning unclear. Formally, Let $B = \{X > Y\} \in \sigma(Y)$. Then

$$\int_B P(X > y)dF_Y(y) = \int_B (1 - P(X \le y))dF_Y(y) = P(Y \in B) - \int_B P(X \le y)dF_Y(y)$$
$$= P(Y \in B) - P(X \le Y, Y \in B) = 0$$

and, analogously, $\int_{B^c} P(X \le y)dF_Y(Y) = 0$. Thus, there is a set $Z \in \sigma(Y)$ with $P(Z) = 0$ and such that for all $y \in B\backslash N$, $P(X > y) = 0$ and for all $y \in B^c\backslash N$, $P(X \le y) = 0$. So we put $a = \sup\{y : y \in B^c\backslash N\}$ and $b = inf\{y : y \in B\backslash N\}$. Then, unless $a = b$, $P(X < a) = 0, P(X \le b) = 1$. Moreover, $B\backslash N \subset [b, \infty)$ and $B^c\backslash N \subset (-\infty, a)$. Then $(a, b) \subset N$ and thus, $P(a < Y < b) = 0$, and also $P(a \le X \le b) = 1$.

Conclusion: If $\{X > Y\} \in \sigma(Y)$, then $P(a \le X \le b) = 1$, and either:

- $a = b$, or
- $P(a < Y < b) = 0$.

(c) Show that if the distribution of Y is continuous and $P(Y \in \text{supp}(X)) > 0$, then $\{X < Y\} \notin \sigma(Y)$.

Proof:

(1) **Assumptions and Definitions:**
- The support of X, $\text{supp}(X)$, is the smallest closed set such that $P(X \in \text{supp}(X)) = 1$.
- Y has a continuous distribution, so for any y, $P(Y = y) = 0$.
- $P(Y \in \text{supp}(X)) > 0$, meaning there exists a set of positive measure where Y can take values in $\text{supp}(X)$.

This is correct.

(2) **Contradiction if $\{X < Y\} \in \sigma(Y)$:**
- If $\{X < Y\} \in \sigma(Y)$, then the event $\{X < Y\}$ depends only on Y. For each realization of $Y = y$, we can determine whether $X < Y$.
- However, since $Y \in \text{supp}(X)$ with positive probability and $P(Y = y) = 0$ (due to the continuity of Y), the comparison $X < Y$ cannot be determined by Y alone. This is because X is independent of Y, and X takes values over its full support.

(3) **Support Argument:**
- For $Y \in \text{supp}(X)$, $P(X = Y) > 0$ for values in the overlap of their supports. Hence, for such y, $P(X < Y)$ cannot depend solely on Y, as X and Y are independent.

Again informal. It can be formalized, but it is easier to use (b) and show a contrapositive. If $\{X < Y\} \in \sigma(Y)$, then Part (b) tells us that there are numbers $a \le b$ such that either $a = b$ or $P(a < Y < b) = 0$ and $\text{supp}(X) \subset [a, b]$. If $a = b$, then $P(Y \in \text{supp}(X)) = 0$ since Y is continuous. If not, then

$$P(Y \in \text{supp}(X)) \le P(a \le Y \le b) = P(a < Y < b) = 0,$$

where the first of the two equalities is by continuity of Y and the second by part (b)

Conclusion: The assumption that $\{X < Y\} \in \sigma(Y)$ contradicts the independence of X and Y. Therefore, $\{X < Y\} \notin \sigma(Y)$.

End of solution