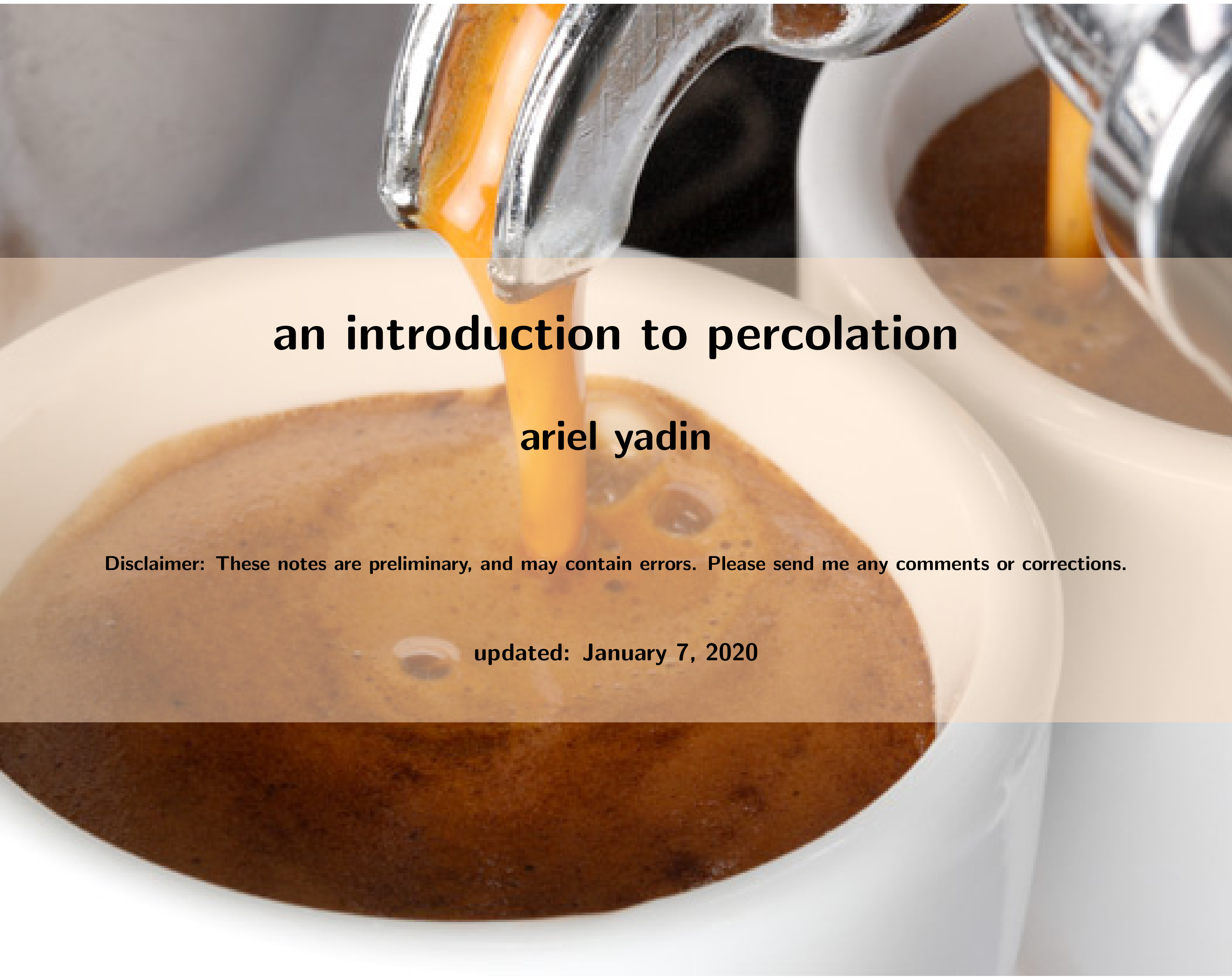

an introduction to percolation

ariel yadin

Disclaimer: These notes are preliminary, and may contain errors. Please send me any comments or corrections.

updated: January 7, 2020

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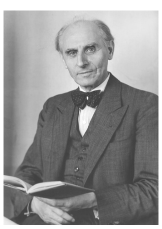
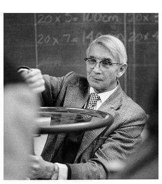
Chapter 1

Introduction

Pierre Curie 1859–1906



Wilhelm Lenz (1888–1957)



Ernst Ising (1900–1998)

1.1 History

In the second half of the 19th century, Pierre Curie experimented with magnetism of iron. When placing an iron block in a magnetic field, sometimes the iron will stay magnetized after removing it from the field, and sometimes the iron will lose the magnetic properties after removal of the field. Curie discovered that this has to do with the temperature; if the temperature is below some critical temperature T_c, then the iron will retain magnetism. If the temperature is higher than T_c then the iron will lose its magnetic properties.

In the 1920's, Lenz gave his student Ising a mathematical model for the above phenomena, and Ising proceeded to study it, solving the one dimensional case in his thesis in 1924. Ising proved that in dimension one, the magnetization is lost in all temperatures, and conjectured that this is also the case in higher dimensions.

This was widely accepted until 1936 (!) when Peierls showed that in dimension 2 and up, there is a phase transition at some finite temperature.

What is percolation?

Let us think of glaciers and ice sheets. As the temperature in the ice rises, electromagnetic bonds between the molecules are becoming less prevalent than the kinetic energy of the molecules, so bounds between the molecules are starting to break. At some point, a lot of the molecules are no longer bound to each other, and they start to move more as individuals than as a structured lattice. The glacier is melting.

This is not exactly what is going on, but it will supply some "real world" motivation for the mathematical model we are about to introduce.

Figure 1.1: Glacier du Trient near the French-Swiss border on the Mont-Blanc massif.

In this course we will deal with a cousin of the so-called Ising model: a model known as *percolation*, first defined by Broadbent and Hammersley in 1957. We will mainly be interested in such phase transition phenomena when some parameter varies (as the temperature in the Ising model).

1.2 Preliminaries - Graphs

We will make use of the structure known as a *graph*:

✓ NOTATION: For a set S we use $\binom{S}{k}$ to denote the set of all subsets of size k in S; *e.g.*

$$\binom{S}{2} = \{\{x, y\} \ : \ x, y \in S, x \neq y\} .$$

Definition 1.2.1 A **graph** G is a pair $G = (V(G), E(G))$, where $V(G)$ is a countable set, and $E(G) \subset \binom{V(G)}{2}$.

The elements of $V(G)$ are called **vertices**. The elements of $E(G)$ are called **edges**. The notation $x \overset{G}{\sim} y$ (sometimes just $x \sim y$ when G is clear from the context) is used for $\{x, y\} \in E(G)$. If $x \sim y$, we say that x is a **neighbor** of y, or that x is **adjacent** to y. If $x \in e \in E(G)$ then the edge e is said to be **incident** to x, and x is incident to e.

The **degree** of a vertex x, denoted $\deg(x) = \deg_G(x)$ is the number of edges incident to x in G.

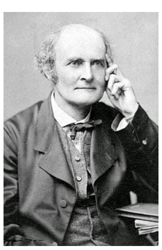
✓ NOTATION: Many times we will use $x \in G$ instead of $x \in V(G)$.

Example 1.2.2

- The complete graph.
- Empty graph on n vertices.
- Cycles.
- $\mathbb{Z}, \mathbb{Z}^2, \mathbb{Z}^d$.
- Regular trees.

△ ▽ △

Example 1.2.3 Cayley graphs of finitely generated groups: Let $G = \langle S \rangle$ be a finitely generated group, with a finite generating set S such that S is symmetric ($S = S^{-1}$). Then, we can equip G with a graph structure $C = C_{G,S}$ by letting $V(C) = G$ and $\{g, h\} \in E(C)$ iff $g^{-1}h \in S$.
S being symmetric implies that this is a graph.
$C_{G,S}$ is called the **Cayley graph** of G with respect to S.
Examples: $\mathbb{Z}^d$, regular trees, cycles, complete graphs.
We will use G and (G, S) to denote the graph $C_{G,S}$. △ ▽ △



Arthur Cayley (1821–1895)

Definition 1.2.4 Let G be a graph. A **path** in G is a sequence $\gamma = (\gamma_0, \gamma_1, \ldots, \gamma_n)$ (with the possibility of $n = \infty$) such that for all j, $\gamma_j \sim \gamma_{j+1}$. γ_0 is the start vertex and γ_n is the end vertex (when $n < \infty$).

The **length** of γ is $|\gamma| = n$.

If γ is a path in G such that γ starts at x and ends at y we write $\gamma : x \to y$.

The notion of a path on a graph gives rise to two important notions: *connectivity* and *graph distance*.

Definition 1.2.5 Let G be a graph. For two vertices $x, y \in G$ define

$$\mathrm{dist}(x,y) = \mathrm{dist}_G(x,y) := \inf \{ |\gamma| \ : \ \gamma : x \to y \} ,$$

where $\inf \emptyset = \infty$.

Exercise 1.1 Show that dist_G defines a metric on G.

(Recall that a metric is a function that satisfies:

- $\rho(x,y) \geq 0$ and $\rho(x,y) = 0$ iff $x = y$.
- $\rho(x,y) = \rho(y,x)$.
- $\rho(x,y) \leq \rho(x,z) + \rho(z,y)$.)

◇◇◇

Definition 1.2.6 Let G be a graph. We say that vertices x and y are **connected** if there exists a path $\gamma : x \to y$ of finite length. That is, if $\mathrm{dist}_G(x,y) < \infty$.

The relation x is connected to y is an equivalence relation, so we can speak of equivalence classes. The equivalence class of a vertex x under this relation is called the **connected component** of x.

If a graph G has only one connected component it is called **connected**. That is, G is connected if for every $x, y \in G$ we have that $x \leftrightarrow y$.

Exercise 1.2 Prove that connectivity is an equivalence relation in any graph. ◇◇◇

✓ All graphs we will consider will have *bounded geometry*. Specifically, we will always assume bounded degree; *i.e.* the degree of any vertex is at most some number d.

✓ NOTATION: For a path in a graph G, or more generally, a sequence of elements from a set S, we use the following "time" notation: If $s = (s_0, s_1, \ldots, s_n, \ldots)$ is a sequence in S (finite of infinite), then $s[t_1, t_2] = (s_{t_1}, s_{t_1+1}, \ldots, s_{t_2})$ for all integers $t_2 \geq t_1 \geq 0$.

1.2.1 Transitive Graphs

If G is a graph, then an **automorphism** of G is a bijection $\varphi : G \to G$ such that $x \sim y$ if and only if $\varphi(x) \sim \varphi(y)$ (that is, φ preserves the graph structure). The set of all automorphisms of a graph G is a group under composition, and is denoted by $\mathrm{Aut}(G)$.

Definition 1.2.7 A graph G is called **transitive** if for every $x, y \in G$ there exists $\varphi_{x,y} \in \mathrm{Aut}(G)$ such that $\varphi_{x,y}(x) = y$. (That is, the group $\mathrm{Aut}(G)$ acts transitively on G.)

Exercise 1.3 Let $G = \langle S \rangle$ be a finitely generated group with symmetric generating set S. Show that the Cayley graph of G is transitive. For every $x, y \in G$ give an example of an automorphism $\varphi_{x,y}$ that maps $x \mapsto y$. ◇◇◇

1.3 Percolation

What is percolation?

Suppose G is a graph. Consider the edges of G, and declare an edge of G "open" with probability p and closed with probability $1 - p$, all edges independent. Consider the (random) sub-graph of G induced by the open edges. This is not necessarily a connected graph.

We will be interested in the connectivity properties of this random subgraph, especially in the existence of infinite connected components.

Let us give a proper probabilistic definition.

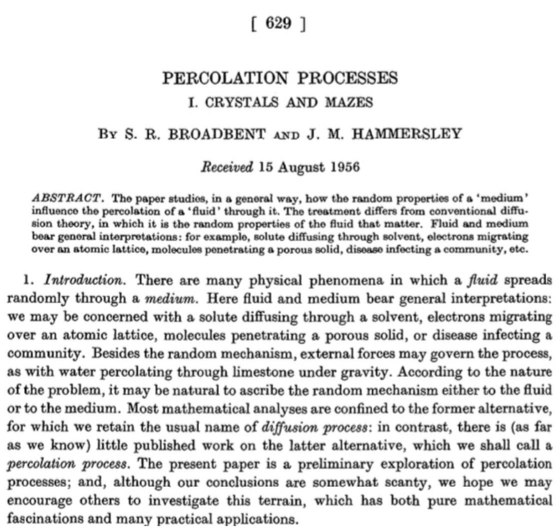
Definition 1.3.1 Let G be a graph. Fix $p \in [0,1]$. Let $(\omega(e))_{e \in E(G)}$ be i.i.d. Bernoulli random variables with mean $\mathbb{P}[\omega(e) = 1] = p$. If $\omega(e) = 1$ we say that e is **open**, otherwise we say that e is **closed**.

Consider the subgraph ω of G whose vertices are $V(\omega) = V(G)$ and edges $E(\omega) = \{e \in E(G) \ : \ \omega(e) = 1\}$.

- ω is called **p-bond-percolation on G**.
- For $x \in G$ denote $\mathcal{C}(x)$ the connected component of x in ω.
- If $y \in \mathcal{C}(x)$ we write $x \leftrightarrow y$ and say that x **is connected to** y. For sets A, B we write $A \leftrightarrow B$ if there exist $a \in A$ and $b \in B$ such that $a \leftrightarrow b$.
- If $|\mathcal{C}(x)| = \infty$ we write $x \leftrightarrow \infty$.

John Hammersley (1920–2004)

✓ This model was first defined by Broadbent and Hammersley in 1957.



[629]

PERCOLATION PROCESSES

I. CRYSTALS AND MAZES

By S. R. BROADBENT and J. M. HAMMERSLEY

Received 15 August 1956

ABSTRACT. The paper studies, in a general way, how the random properties of a 'medium' influence the percolation of a 'fluid' through it. The treatment differs from conventional diffusion theory, in which it is the random properties of the fluid that matter. Fluid and medium bear general interpretations: for example, solute diffusing through solvent, electrons migrating over an atomic lattice, molecules penetrating a porous solid, disease infecting a community, etc.

1. *Introduction.* There are many physical phenomena in which a *fluid* spreads randomly through a *medium*. Here fluid and medium bear general interpretations: we may be concerned with a solute diffusing through a solvent, electrons migrating over an atomic lattice, molecules penetrating a porous solid, or disease infecting a community. Besides the random mechanism, external forces may govern the process, as with water percolating through limestone under gravity. According to the nature of the problem, it may be natural to ascribe the random mechanism either to the fluid or to the medium. Most mathematical analyses are confined to the former alternative, for which we retain the usual name of *diffusion process*: in contrast, there is (as far as we know) little published work on the latter alternative, which we shall call a *percolation process*. The present paper is a preliminary exploration of percolation processes; and, although our conclusions are somewhat scanty, we hope we may encourage others to investigate this terrain, which has both pure mathematical fascinations and many practical applications.

Figure 1.2: The title page of the original paper by Broadbent & Hammersley.

1.4 Kolmogorov's $0-1$ Law

Let us briefly recall the measure theory of infinite product spaces.

The σ-algebra for the space containing $(\omega(e))_{e\in E(G)}$ is the σ-algebra generated by the cylinder sets:

If $E \subset E(G), |E| < \infty$ is a finite subset of edges, and $\eta \in \{0,1\}^{E(G)}$ is a $0,1$-vector indexed by $E(G)$, then the cylinder around E at η is the set

$$C_{\eta,E} := \left\{ \omega \in \{0,1\}^{E(G)} \ : \ \forall \ e \in E \ , \ \omega(e) = \eta(e) \right\}.$$

These are the basic measurable sets.

The σ-algebra $\mathcal{F}$ for the space is then the σ algebra generated by all cylinders:

$$\mathcal{F} = \sigma(C_{\eta,E} \ : \ \eta \in \{0,1\}^{E(G)} \ , \ E \subset E(G), |E| < \infty).$$

For fixed finite subset $E \subset E(G), |E| < \infty$, let

$$\mathcal{F}_E = \sigma(\omega(e) \ : \ e \in E) = \sigma \left\{ C_{\eta,E} \ : \ \eta \in \{0,1\}^{E(G)} \right\}.$$

This is the information carried by the edges in E. Of course $\mathcal{F}_E \subset \mathcal{F}$.

Perhaps the most useful property of $\mathcal{F}$ is that the cylinder sets are dense in $\mathcal{F}$: that is, for any event $A \in \mathcal{F}$, and any $\varepsilon > 0$, there exists a finite $E \subset E(G), |E| < \infty$ and an event $B \in \mathcal{F}_E$ such that $\mathbb{P}[A \triangle B] < \varepsilon$.

$\mathcal{F}$ is defined to be the smallest σ-algebra containing the cylinder sets, but does $\mathcal{F}$ contain more than cylinders? In fact, it does.

For a finite subset $E \subset E(G), |E| < \infty$, one can also consider the events that do not depend on the edges in E:

$$\mathcal{T}_E = \sigma(\omega(e) \ : \ e \notin E).$$

And define the **tail**-σ-algebra

$$\mathcal{T} = \bigcap_E \mathcal{T}_E.$$

These are events the do not depend on a finite configuration of edges; in other words, for a tail event $A \in \mathcal{T}$, $\omega \in A$ if and only if for any finite $E \subset E(G), |E| < \infty$ and any $\eta \in \{0,1\}^{E(G)}$ such that $\eta(e) = \omega(e)$ for all $e \notin E$, also $\eta \in A$.

Exercise 1.4 Let ω be p-bond-percolation on a graph G.

Show that the event that $|\mathcal{C}(x)| = \infty$ is measurable.

Show that the event that there exists an infinite component in ω is measurable.

Show that this event is a tail event. ◇◇◇

Theorem 1.4.1 **(Kolmogorov's $0,1$ Law)** Let $\omega(e)_e$ be i.i.d. random variables and let A be a tail event. Then, $\mathbb{P}[A] \in \{0,1\}$.

Andrey Kolmogorov (1903–1987)

Proof. Let A be a tail event. So $A \in \mathcal{T}_E$ for all finite $E \subset E(G), |E| < \infty$. Let $(A_n)_n$ be a sequence of events such that $\mathbb{P}[A \triangle A_n] \to 0$, and $A_n \in \mathcal{F}_n$ for all n, where $\mathcal{F}_n = \mathcal{F}_{E_n}$ for some finite subset $E_n \subset E(G), |E_n| < \infty$. Note that

$$\begin{aligned}|\,\mathbb{P}[A_n] - \mathbb{P}[A]| &= |\,\mathbb{P}[A_n \setminus A] + \mathbb{P}[A \cap A_n] - \mathbb{P}[A]| = |\,\mathbb{P}[A_n \setminus A] - \mathbb{P}[A \setminus A_n]| \\ &\leq \mathbb{P}[A_n \setminus A] + \mathbb{P}[A \setminus A_n] \to 0.\end{aligned}$$

Since $\mathcal{F}_n$ is independent of $\mathcal{T}_{E_n}$ we have that for all n, $\mathbb{P}[A \cap A_n] = \mathbb{P}[A]\,\mathbb{P}[A_n] \to \mathbb{P}[A]^2$. On the other hand,

$$\mathbb{P}[A] \leq \mathbb{P}[A \cup A_n] = \mathbb{P}[A \cap A_n] + \mathbb{P}[A \triangle A_n] \to \mathbb{P}[A]^2.$$

Since $\mathbb{P}[A] \in [0,1]$ we must have that $\mathbb{P}[A] = \mathbb{P}[A]^2$ and so $\mathbb{P}[A] \in \{0,1\}$. $\square$

Corollary 1.4.2 For a graph G define $\Theta(p) = \Theta_G(p)$ to be the probability that p-bond percolation on G contains an infinite component. Then, $\Theta(p) \in \{0,1\}$.

Chapter 2

Basic Properties

2.1 Monotonicity

Recall that the sample space for percolation on G is $\{0,1\}^{E(G)}$. There is a natural partial order on elements of this space: $\omega \leq \eta$ if $\omega(e) \leq \eta(e)$ for all $e \in E(G)$. In other words, a configuration η is larger than ω if any edge open in ω is also open in η.

Definition 2.1.1 Let A be an event in percolation on G. We say that A is **increasing** if $\omega \leq \eta$ and $\omega \in A$ imply that $\eta \in A$; that is, opening more edges for configurations in A remains in A.

We say that A is **decreasing** if $\omega \leq \eta$ and $\eta \in A$ implies that $\omega \in A$; that is closing edges remains in A.

Exercise 2.1 Show that $\{x \leftrightarrow \infty\}$ is an increasing event.

Show that $\{x \leftrightarrow y\}$ is an increasing event.

Show that A is increasing if and only if A^c is decreasing.

Show that the union of increasing events is increasing.

Show that the intersection of increasing events is increasing. ◇◇◇

✓ One can also define increasing (respectively, decreasing) functions on

$\{0,1\}^{E(G)}$.

Consider the following procedure to generate p-percolation on a graph G: Let $(U_e)_{e\in E(G)}$ be i.i.d. random variables indexed by the edges of G, each distributed uniformly on $[0,1]$. For any $p \in [0,1]$, let $\Omega_p(e) = 1$ if $U_e \leq p$.

Then, Ω_p is a p-percolation process on G. Moreover, this couples all percolation processes on G in one space, with the property that for $p \leq q$, if an edge e is open in Ω_p then it is open in Ω_q.

✓ Notation: Henceforth, we will use $\mathbb{P}, \mathbb{E}$ to denote the probability measure and expectation on the space of the above coupling (that is, of the i.i.d. sequence of uniform random variables). In this space, we use Ω_p to denote the percolation cluster induced by the edges e with $U_e \leq p$. $\mathbb{P}_p, \mathbb{E}_p$ denote the measure and expectation on the space of p-percolation. If it is not clear from the context, we add super-scripts $\mathbb{P}_p^G, \mathbb{E}_p^G$ to stress the graph on which percolation is performed.

Lemma 2.1.2 Let A be an increasing event on $\{0,1\}^{E(G)}$. Let B be a decreasing event. Then, for all $p \leq q$,

$$\mathbb{P}_p[A] \leq \mathbb{P}_q[A] \qquad \text{and} \qquad \mathbb{P}_p[B] \geq \mathbb{P}_q[B].$$

Proof. Under the natural coupling with uniform random variables, recall that $\Omega_p(e) = 1$ if $U_e \leq p$. Since $\Omega_p \leq \Omega_q$ for $p \leq q$, we get that if $\Omega_p \in A$ then $\Omega_q \in A$. So

$$\mathbb{P}_p[A] = \mathbb{P}[\Omega_p \in A] \leq \mathbb{P}[\Omega_q \in A] = \mathbb{P}_q[A].$$

The proof for B follows by noticing that B^c is an increasing event. □

Monotonicity and the $0,1$ law combined give:

Corollary 2.1.3 Let $\Theta_G(p)$ denote the probability that there exists an infinite component. Then, there exists $p_c = p_c(G) \in [0,1]$ such that for all $p < p_c$, $\Theta_G(p) = 0$ and for all $p > p_c$, $\Theta_G(p) = 1$.

Proof. Just define $p_c = \sup\{p \,:\, \Theta_G(p) = 0\}$. □

The structure of Θ_G is almost fully understood. Two main questions remain:

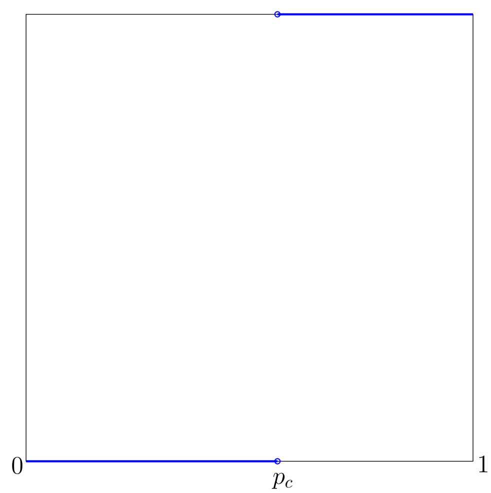


Figure 2.1: The quite boring graph of Θ_G

- For a graph G what is the transition point $p_c(G)$? When is $0 < p_c(G) < 1$?
- For a graph G, does percolation percolate at criticality? That is, what is $\Theta_G(p_c)$?

The second question is perhaps the most important open question in percolation theory.

Exercise 2.2 Show that $p_c(\mathbb{Z}) = 1$. ◇◇◇

2.2 Translation Invariance

Last chapter we used Kolmogorov's $0, 1$ Law to show that $\Theta(p) \in \{0, 1\}$. We reprove this for transitive graphs.

For a graph G, we say that an event $A \in \mathcal{F}$ is **translation invariant** if $\varphi A = A$ for all $\varphi \in \mathrm{Aut}(G)$. Here $\varphi A = \{\varphi\omega : \omega \in A\}$ where $(\varphi\omega)(e) = \omega(\varphi^{-1}(e))$ and $\varphi(\{x, y\}) = \{\varphi(x), \varphi(y)\}$.

Exercise 2.3 Show that the event that there exists an infinite component is translation invariant. ◇◇◇

Lemma 2.2.1 Let G be an infinite transitive graph. If A is a translation invariant event then $\mathbb{P}[A] \in \{0,1\}$.

Proof. Transitivity is used in the following

Exercise 2.4 Let G be an infinite transitive graph, and let $E \subset E(G), |E| < \infty$ be some finite subset. Then, there exists $\varphi \in \mathrm{Aut}(G)$ such that $\varphi E \cap E = \emptyset$. ◇◇◇

Given this exercise, let A be translation invariant. Let $(A_n)_n$ be a sequence of events $A_n \in \mathcal{F}_n$ where $\mathcal{F}_n = \mathcal{F}_{E_n}$ for finite subsets $E_n \subset E(G), |E_n| < \infty$, and such that $\mathbb{P}[A \triangle A_n] \to 0$.

This tells us that in some sense we can replace A by A_n without losing much. Indeed, for any $\varphi \in \mathrm{Aut}(G)$,

$$\begin{aligned}\mathbb{P}[A \cap \varphi A] - \mathbb{P}[A_n \cap \varphi A_n] &\leq \mathbb{P}[(A \cap \varphi A) \setminus (A_n \cap \varphi A_n)] \leq \mathbb{P}[A \setminus A_n] + \mathbb{P}[\varphi(A \setminus A_n)] \\ &= 2\,\mathbb{P}[A \setminus A_n] \to 0.\end{aligned}$$

For every n let $\varphi_n \in \mathrm{Aut}(G)$ be such that $\varphi_n E_n \cap E_n = \emptyset$. Since $A_n \in \mathcal{F}_n$, we get that A_n is independent of $\varphi_n A_n$ for all n. Since $A = \varphi_n A$ for all n,

$$\begin{aligned}\mathbb{P}[A] = \mathbb{P}[A \cap \varphi_n A] &\leq \mathbb{P}[A_n \cap \varphi_n A_n] + 2\,\mathbb{P}[A \triangle A_n] \\ &= \mathbb{P}[A_n]\,\mathbb{P}[\varphi_n A_n] + 2\,\mathbb{P}[A \triangle A_n] \to \mathbb{P}[A]^2.\end{aligned}$$

So $\mathbb{P}[A] = \mathbb{P}[A]^2$ and thus must be in $\{0,1\}$. □

This provides another proof of Corollary 1.4.2 for transitive graphs.

Chapter 3

Percolation Probability ($\theta(p)$)

3.1 Cluster at a Specific Vertex

It will be more convenient to study the cluster at a specific vertex, $\mathcal{C}(x)$.

Definition 3.1.1 For percolation on an infinite connected graph G, define the function $\theta_{G,x}(p) = \mathbb{P}_p[x \leftrightarrow \infty]$. If G is a transitive graph, then write $\theta_G(p) = \theta_{G,x}(p)$, since this latter function does not depend on x.

Note that $\theta_{G,x}(p)$ is monotone non-decreasing, $\theta_{G,x}(0) = 0$ and $\theta_{G,x}(1) = 1$ (if G is an infinite connected graph). So it would be natural to define

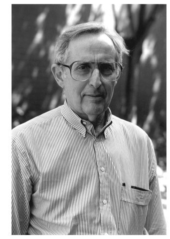
$$p_c(G,x) = \sup \{p \ : \ \theta_{G,x}(p) = 0\} .$$

We will see that actually $p_c(G,x) = p_c(G)$ and does not depend on x.

In fact, we will see that the following are equivalent for $p \in [0,1]$:

- $\Theta_G(p) = 1$.
- $\theta_{G,x}(p) > 0$ for some x.
- $\theta_{G,x}(p) > 0$ for all x.

3.2 Harris' Inequality (FKG)



Ted Harris (1919–2005)

In 1960 Harris showed that $\Theta_{\mathbb{Z}^2}(1/2) = 0$ (so $p_c(\mathbb{Z}^2) \geq 1/2$). In his proof he introduced a correlation inequality, that in words states that any two increasing functions of percolation configurations have positive covariance. This inequality has also come to be commonly known by the FKG inequality, for Fortuin, Kasteleyn and Ginibre who proved some generalizations of it for the non-independent case.

In order to state Harris' inequality properly, we would like the concept of increasing functions on $\{0,1\}^{E(G)}$.

Definition 3.2.1 A function $X : \{0,1\}^{E(G)} \to \mathbb{R}$ is called **increasing** if $X(\omega) \leq X(\eta)$ for all $\omega \leq \eta$. X is called **decreasing** if $X(\omega) \geq X(\eta)$ for all $\omega \leq \eta$.

Exercise 3.1 Let A be an event. Show that the function $\mathbf{1}_A$ is increasing if and only if A is an increasing event. $\diamond\diamond\diamond$

✓ Any measurable function $X : \{0,1\}^{E(G)} \to \mathbb{R}$ is a random variable, so we can speak of expectation, moments, etc.

Exercise 3.2 Prove that if X is an increasing integrable random variable then for any $p \leq q$ we have $\mathbb{E}_p[X] \leq \mathbb{E}_q[X]$. $\diamond\diamond\diamond$

We turn to Harris' Lemma:

Lemma 3.2.2 For percolation on a graph G let X, Y be two increasing random variables in L^2 (*i.e.* $\mathbb{E}[X^2], \mathbb{E}[Y^2] < \infty$). For any p,

$$\mathbb{E}_p[XY] \geq \mathbb{E}_p[X] \cdot \mathbb{E}_p[Y].$$

Exercise 3.3 For percolation on G, let X, Y be decreasing random variables in L^2. Let Z be an increasing random variable in L^2. Show that for any p,

- $\mathbb{E}_p[XZ] \leq \mathbb{E}_p[X] \cdot \mathbb{E}_p[Z]$.

- $\mathbb{E}_p[XY] \geq \mathbb{E}_p[X] \cdot \mathbb{E}_p[Y]$.

◇◇◇

Proof of Lemma 3.2.2. First, note that

$$\mathbb{E}[XY] \geq \mathbb{E}[X]\,\mathbb{E}[Y] \iff \mathbb{E}[(X - \mathbb{E}[X])(Y - \mathbb{E}[Y])] \geq 0,$$

and that X is increasing if and only if $X - \mathbb{E}[X]$ is increasing. So we can assume without loss of generality that $\mathbb{E}[X] = 0$ and $\mathbb{E}[Y] = 0$.

Next, we prove the lemma in the case where $X, Y \in \mathcal{F}_E$ for some finite $E \subset E(G), |E| < \infty$. This is done by induction on $|E|$.

Base: $|E| = 1$. Suppose $E = \{e\}$. In this case, X, Y are functions of $\omega(e) = 1$ or $\omega(e) = 0$. That is, X is determined by two numbers $X_0, X_1 \in \mathbb{R}$:

$$\forall\ \omega \in \{0,1\}^{E(G)} X(\omega) = \mathbf{1}_{\{\omega(e)=0\}} \cdot X_0 + \mathbf{1}_{\{\omega(e)=1\}} \cdot X_1.$$

Since $\mathbb{E}[X] = 0$, we have that $0 = \mathbb{E}[X] = pX_1 + (1-p)X_0$, so $X_1 = -\frac{1-p}{p}X_0$. Since X is increasing, $X_0 \leq X_1$. So it must be that $X_0 \leq 0$. Similarly, for Y with $Y(\omega) = \mathbf{1}_{\{\omega(e)=0\}} \cdot Y_0 + \mathbf{1}_{\{\omega(e)=1\}} \cdot Y_1$ we have $Y_1 = -\frac{1-p}{p}Y_0$ and $Y_0 \leq 0$. Now,

$$\mathbb{E}[XY] = pX_1Y_1 + (1-p)X_0Y_0 = X_0Y_0 \cdot \left(\frac{(1-p)^2}{p} + 1 - p\right) = X_0Y_0 \cdot \frac{1-p}{p} \geq 0.$$

Induction step: $|E| = n + 1$. Assume that $E = \{e_0, e_1, \ldots, e_n\}$. Let $E' = \{e_1, \ldots, e_n\}$.

Let $\eta \in \{0,1\}^{E'}$. For $\omega \in \{0,1\}^{E(G)}$ let ω_η be defined by $\omega_\eta(e) = \omega(e)$ if $e \notin E'$ and $\omega_\eta(e) = \eta(e)$ if $e \in E'$. Consider the random variables

$$X_\eta(\omega) = X(\omega_\eta) \qquad \text{and} \qquad Y_\eta(\omega) = Y(\omega_\eta).$$

Then, $X_\eta, Y_\eta \in \mathcal{F}_{\{e_0\}}$ and are increasing. So by the $|E| = 1$ case,

$$\mathbb{E}[X_\eta Y_\eta] \geq \mathbb{E}[X_\eta] \cdot \mathbb{E}[Y_\eta].$$

This holds for any choice of $\eta \in \{0,1\}^{E'}$. Thus, a.s.

$$\mathbb{E}[XY|\mathcal{F}_{E'}] \geq \mathbb{E}[X|\mathcal{F}_{E'}] \cdot \mathbb{E}[Y|\mathcal{F}_{E'}].$$

Now, consider the random variables

$$X' = \mathbb{E}[X|\mathcal{F}_{E'}] \qquad \text{and} \qquad Y' = \mathbb{E}[Y|\mathcal{F}_{E'}].$$

Note that X', Y' are increasing (because X, Y are). Since $|E'| = n$, by induction,

$$\begin{aligned}\mathbb{E}[XY] &= \mathbb{E}\,\mathbb{E}[XY|\mathcal{F}_{E'}] \geq \mathbb{E}\left[\mathbb{E}[X|\mathcal{F}_{E'}] \cdot \mathbb{E}[Y|\mathcal{F}_{E'}]\right] \\ &\geq \mathbb{E}\,\mathbb{E}[X|\mathcal{F}_{E'}] \cdot \mathbb{E}\,\mathbb{E}[Y|\mathcal{F}_{E'}] = \mathbb{E}[X] \cdot \mathbb{E}[Y].\end{aligned}$$

This proves Harris' Lemma for random variables that depend on finitely many edges.

To extend to infinitely many edges we use the (L^2-) Martingale Convergence Theorem:

Theorem 3.2.3 (Martingale Convergence Theorem) Let $(M_n)_n$ be a L^2-bounded martingale (that is, such that $\sup_n \mathbb{E}[M_n^2] < \infty$). Then, $M_n \to M$ a.s. and in L^2 for some M in L^2.

A proof, and more on martingales, can be found in *Probability: Theory and Examples* by Rick Durrett, Chapter 5.

Now, back to Harris' Lemma: Let $e_1, e_2, \ldots,$ be some ordering of the edges in $E(G)$. Let $\mathcal{F}_n = \mathcal{F}_{\{e_1,\ldots,e_n\}}$. If X is in L^2 then $X_n := \mathbb{E}[X|\mathcal{F}_n]$ is a martingale (known as the *information exposure martingale*). Similarly for $Y_n := \mathbb{E}[Y|\mathcal{F}_n]$. Now, point-wise, $X_n \to X$ and $Y_n \to Y$. Since $(X_n)_n, (Y_n)_n$ converge to a limit a.s. and in L^2 by the Martingale Convergence Theorem, these limits must be X and Y. That is,

$$\mathbb{E}[(X - X_n)^2] \to 0 \qquad \text{and} \qquad \mathbb{E}[(Y - Y_n)^2] \to 0.$$

An application of Cauchy-Schwarz gives,

$$\begin{aligned}\mathbb{E}[|XY - X_nY_n|] &\leq \mathbb{E}[|X| \cdot |Y - Y_n|] + \mathbb{E}[|Y_n| \cdot |X - X_n|] \\ &\leq \sqrt{\mathbb{E}[X^2] \cdot \mathbb{E}[(Y - Y_n)^2]} + \sqrt{\mathbb{E}[Y_n^2] \cdot \mathbb{E}[(X - X_n)^2]} \to 0.\end{aligned}$$

For every n, X_n, Y_n are increasing. So by the finite case of Harris' Lemma, $\mathbb{E}[X_nY_n] \geq \mathbb{E}[X_n] \cdot \mathbb{E}[Y_n]$. Hence,

$$\mathbb{E}[XY] = \lim_{n\to\infty} \mathbb{E}[X_nY_n] \geq \lim_{n\to\infty} \mathbb{E}[X_n] \cdot \mathbb{E}[Y_n] = \mathbb{E}[X] \cdot \mathbb{E}[Y].$$

□

Exercise 3.4 Replace the use of the martingale convergence theorem in the previous proof with the following straightforward argument:

Let $X : \{0,1\}^{E(G)} \to \mathbb{R}$ be a random variable in L^2. Let $(E_n)_n$ be a sequence of subsets in $E(G)$. Assume that $\bigcup_n E_n = E(G)$. For each n define $X_n = \mathbb{E}_p[X \mid \mathcal{F}_{E_n}]$.

Show that $X_n \to X$ a.s.

Show that $X_n \to X$ in L^2.

Show that if X is increasing then so is X_n, for any n. ◇◇◇

3.3 Infinite Cluster at a Vertex

We now use Harris' Inequality to prove

Theorem 3.3.1 Let G be an infinite connected graph. The following are equivalent.

- $\Theta_G(p) = 1$.
- There exists x such that $\theta_{G,x}(p) > 0$.
- For all x, $\theta_{G,x}(p) > 0$.

Proof. We will show that for any x, $\Theta_G(p) = 1$ if and only if $\theta_{G,x}(p) > 0$.

Note that

$$\Theta_G(p) \leq \sum_z \theta_{G,z}(p).$$

So if $\Theta_G(p) = 1$ then there exists z such that $\theta_{G,z}(p) > 0$. Since G is connected, the event $\{x \leftrightarrow z\}$ has positive probability. Since this event is increasing, Harris' Lemma gives

$$\theta_{G,x}(p) = \mathbb{P}_p[x \leftrightarrow \infty] \geq \mathbb{P}_p[x \leftrightarrow z, z \leftrightarrow \infty] \geq \mathbb{P}_p[x \leftrightarrow z] \cdot \theta_{G,z}(p) > 0.$$

One the other hand, if $\theta_{G,x}(p) > 0$ then

$$\Theta_G(p) = \mathbb{P}_p[\exists\, z \ : \ z \leftrightarrow \infty] \geq \mathbb{P}_p[x \leftrightarrow \infty] > 0,$$

so the 0, 1 law gives that $\Theta_G(p) = 1$. $\square$

Corollary 3.3.2 Let G be an infinite connected graph. Then, for any $p < p_c(G)$ and any x we have that $\theta_{G,x}(p) = 0$. For any $p > p_c$ and any x we have that $\theta_{G,x}(p) > 0$. Thus,

$$\begin{aligned} p_c(G) &= \sup\{p \ : \ \Theta_G(p) = 0\} = \inf\{p \ : \ \Theta_G(p) = 1\} \\ &= \sup\{p \ : \ \theta_{G,x}(p) = 0\} = \inf\{p \ : \ \theta_{G,x}(p) > 0\}. \end{aligned}$$

Thus we are left with the two interesting and basic questions:

- What is $p_c(G)$? When is $p_c(G) < 1$?
- What is $\Theta_G(p_c(G))$?

Exercise 3.5 Let H be a subgraph of G. Show that $p_c(H) \geq p_c(G)$. $\diamond\diamond\diamond$

3.4 One-dimensional graphs

Definition 3.4.1 Let o be a vertex in a graph G. A subset of edges $A \subset E(G)$ is called a **cutset** for o if by removing A from G the vertex o is left in a finite connected component. Equivalently, any infinite simple path started at o must cross A.

Definition 3.4.2 A graph G is **one-dimensional** if there exists a vertex $o \in G$ and a sequence of pairwise disjoint cutsets $(A_n)_n$ for o such that $\sup_n |A_n| < \infty$.

Exercise 3.6 Show that $\mathbb{Z}$ is one-dimensional.

Show that the "ladder graph" $\mathbb{Z} \times \{0, 1\}$ is one-dimensional. $\diamond\diamond\diamond$

Exercise 3.7 Let G be a finitely generated group. Assume that H is a finite index subgroup of G, $[G : H] < \infty$, such that $H \cong \mathbb{Z}$. Show that any Cayley graph of G is one-dimensional. $\diamond\diamond\diamond$

Theorem 3.4.3 If G is a one-dimensional graph then $p_c(G) = 1$.

Proof. Let o be a vertex and $(A_n)_n$ a sequence of pairwise disjoint cutsets such that $|A_n| \leq M$ for some constant $M > 0$.

Let $\mathcal{E}_n$ be the event that all edges in A_n are closed. Note that $\mathbb{P}_p[\mathcal{E}_n] \geq (1-p)^M$. Moreover, since the cutsets $(A_n)_n$ are pairwise disjoint, the events $(\mathcal{E}_n)_n$ are independent. Thus, as long as $p < 1$,

$$\mathbb{P}_p[\cap_{k\leq n}(\mathcal{E}_k)^c] \leq (1-(1-p)^M)^n \to 0.$$

That is, the event $\bigcup_n \mathcal{E}_n$ has probability 1. Consider this event. It states that there exists some cutset A_n such that ell edges in A_n are closed. That is, the event $\bigcup_n \mathcal{E}_n$ implies the event that there exists n for which $\mathcal{C}(o)$ is in the subgraph of G with A_n removed. Specifically, $\bigcup_n \mathcal{E}_n \subset \{|\mathcal{C}(o)| < \infty\}$, so $\mathbb{P}_p[|\mathcal{C}(o)| < \infty] = 1$ for all $p < 1$. $\square$

Chapter 4

Phase transition in $\mathbb{Z}^d$

4.1 Peierls' Argument

Rudolf Peierls (1907–1995)

The fact that $p_c(\mathbb{Z}) = 1$ is related to the fact that the Ising model does not retain magnetization at any finite temperature. Peierls originally used the following type of argument to show that the Ising model has non-trivial phase transition in dimension 2 and up, and this is related to the fact that $0 < p_c(\mathbb{Z}^d) < 1$ for all $d \geq 2$. We will use Peierls' argument to prove this.

Theorem 4.1.1 Let G be an infinite connected graph with degrees bounded by D. Then,

$$p_c(G) \geq \frac{1}{D-1}.$$

Proof. A **self-avoiding path** in G started at x is a path γ such that $\gamma_0 = x$ and $\gamma_i \neq \gamma_j$ for all $i \neq j$. Let S_n be the set of self-avoiding paths in G started at x of length n, and let $\mu_n = |S_n|$. Let $\mu = \limsup_n (\mu_n)^{1/n}$.

Note that since D is the maximal degree in G, and since a self-avoiding path cannot backtrack over the last edge it passed through, $\mu_n \leq D(D-1)^{n-1}$ and $\mu \leq D-1$. So it suffices to show that $p_c(G) \geq \mu^{-1}$.

Note that for any path $\gamma \in S_n$, since γ is self-avoiding, γ passes through n different edges, so

$$\mathbb{P}_p[\gamma \subset \mathcal{C}(x)] = \mathbb{P}_p[\gamma \text{ is open }] = p^n.$$

Now,

$$\mathbb{P}_p[\exists\ \gamma \in S_n\ :\ \gamma \subset \mathcal{C}(x)] \leq \sum_{\gamma \in S_n} \mathbb{P}_p[\gamma \subset \mathcal{C}(x)] = \mu_n p^n.$$

Now, the event that $|\mathcal{C}(x)| = \infty$ is the event that $\mathcal{C}(x)$ contains an infinite self-avoiding path started at x. That is, $|\mathcal{C}(x)| = \infty$ implies that for every n, there exists $\gamma \in S_n$ such that $\gamma \subset \mathcal{C}(x)$. Thus,

$$\mathbb{P}_p[x \leftrightarrow \infty] = \lim_{n\to\infty} \mathbb{P}_p[\exists\ \gamma \in S_n\ :\ \gamma \subset \mathcal{C}(x)] \leq \lim_{n\to\infty} (\mu_n^{1/n} p)^n.$$

If $p < \mu^{-1}$ then this last limit is 0. So $\theta_{G,x}(p) = 0$ for all $p < \mu^{-1}$, which implies that $p_c(G) \geq \mu^{-1}$. □

Example 4.1.2 Let $\mathbb{T}_d$ be the d-regular tree. Peierls' argument gives that $p_c(\mathbb{T}_d) \geq \frac{1}{d-1}$.

Later we will see that $p_c(\mathbb{T}_d) = \frac{1}{d-1}$. △▽△

4.2 Upper Bound via Cut-Sets

Let G be an infinite connected graph. Let $x \in G$ be some fixed vertex. Recall that a **cut-set** is a set Π of edges such that any infinite self-avoiding path started at x must pass through an edge of Π. A **minimal cut-set** is a cut-set Π such that for any $e \in \Pi$, the set $\Pi \setminus \{e\}$ is *not* a cut-set (*i.e.* Π is minimal with respect to inclusion).

Exercise 4.1 Show that any finite cut-set must contain a minimal cut-set.
◇◇◇

Proposition 4.2.1 Let G be an infinite connected graph and let $x \in G$. For percolation on G, $x \leftrightarrow \infty$ if and only if for every finite minimal cut-set Π it holds that Π contains at least one open edge.

In other words, $\mathcal{C}(x)$ is finite if and only if there exists a finite cut-set Π such that all edges in Π are closed.

Proof. If $\mathcal{C}(x)$ is finite, then the set $\{\{y,z\} \in E(G)\ :\ y \in \mathcal{C}(x), z \notin \mathcal{C}(x)\}$ is a finite cut-set. All edges in this cut-set must be closed because one of their endpoints is not in $\mathcal{C}(x)$.

If $\mathcal{C}(x)$ is infinite, then there exists an infinite self-avoiding path γ starting at x, whose edges are all open. Let Π be any finite cut-set. Then γ must pass through some edge of Π, so Π must contain an open edge. $\square$

Proposition 4.2.2 Let G be an infinite connected graph, and let $x \in G$. Let C_n be the set of minimal cut-sets of size n. If there exist n_0 and $M > 1$ such that $|C_n| \leq M^n$ for all $n > n_0$, then $p_c(G) \leq \frac{M-1}{M}$.

Proof. Take $p > \frac{M-1}{M}$ so that $M(1-p) < 1$. If $\Pi \in C_n$ then the probability that all eges of Π are closed is $(1-p)^n$. Thus, there exists N large enough so that for

$$A := \exists\ n > N \ : \ \exists\ \Pi \in C_n \ : \ \text{ all edges in } \Pi \text{ are closed}$$

we have $\mathbb{P}_p[A] \leq \sum_{n>N} M^n(1-p)^n \leq \frac{1}{2}$.

Now, let $S = \{\Pi \in C_n \ : \ n \leq N\}$ and let $E = \bigcup_{\Pi \in S} S$.

Let B be the event that all edges in E are open. Since E is finite, $\mathbb{P}_p[B] > 0$. By FKG,

$$\mathbb{P}_p[B \cap A^c] \geq \mathbb{P}_p[B] \cdot (1 - \mathbb{P}_p[A]) \geq \frac{1}{2}\,\mathbb{P}_p[B] > 0.$$

Now, if there exists a minimal cut-set Π such that all the edges of Π are closed, then either $|\Pi| \leq n$ and so B does not occur, or A occurs. Thus, $\{x \not\leftrightarrow \infty\} \subset B^c \cup A$, which implies $\{x \leftrightarrow \infty\} \supset B \cap A^c$. So

$$\theta_{G,x}(p) = \mathbb{P}_p[x \leftrightarrow \infty] \geq \frac{1}{2}\,\mathbb{P}_p[B] > 0,$$

and so $p_c(G) \leq p$.

Since this holds for all $p > \frac{M-1}{M}$ we get that $p_c(G) \leq \frac{M-1}{M}$. $\square$

4.3 Duality

For $\mathbb{Z}^d$, we know that $p_c(\mathbb{Z}^d) \leq p_c(\mathbb{Z}^2)$ for all $d \geq 2$. So in order to show that $\mathbb{Z}^d$ has a non-trivial phase transition it suffices to prove that $p_c(\mathbb{Z}^2) < 1$.

The main tool to show this is *duality*, which is due to the fact that $\mathbb{Z}^2$ is a *planar graph*. This dual structure will help us count the number of minimal cut-sets of a given size.

Suppose G is a planar graph. Then, we can speak about *faces*. Each edge is adjacent to exactly two faces (Euler's formula). So we can define a dual graph $\hat{G}$ whose vertex set is the set of faces of G, and to each edge $e \in E(G)$ we have a corresponding edge $\hat{e} \in E(\hat{G})$ which is the edge connecting the faces adjacent to e.

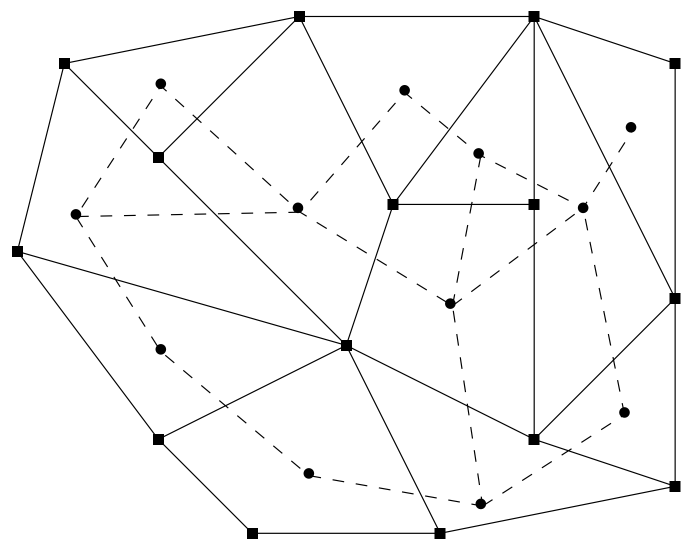


Figure 4.1: A planar graph (vertices are squares, edges are solid lines) and it's dual (vertices are disks, edges are dashed lines).

For $\mathbb{Z}^2$ the dual structure is special: $\widehat{\mathbb{Z}^2}$ is isomorphic to $\mathbb{Z}^2$ itself. Indeed, every face of $\mathbb{Z}^2$ can be identified with the point at it's center, so the vertices of $\widehat{\mathbb{Z}^2}$ are $(\frac{1}{2}, \frac{1}{2}) + \mathbb{Z}^2$. For an edge $e \in E(\mathbb{Z}^2)$ the dual edge is just the edge of $(\frac{1}{2}, \frac{1}{2}) + \mathbb{Z}^2$ that crosses the edge e.

The self duality of $\mathbb{Z}^2$ provides a natural coupling of percolation on the two graphs. For an edge $\hat{e} \in E(\widehat{\mathbb{Z}^2})$ we declare $\hat{e}$ open if and only if e is closed. So p-percolation on $\mathbb{Z}^2$ is coupled to $(1-p)$-percolation on $\widehat{\mathbb{Z}^2}$.

A self-avoiding polygon in a graph G is a finite path γ such that $\gamma_0 = \gamma_n = x$ and $\gamma_j \neq \gamma_i$ for $0 \leq j \neq i < n$, where n is the length of γ; that is, a path that visits every vertex once, except for the initial vertex with coincides with the terminal vertex.

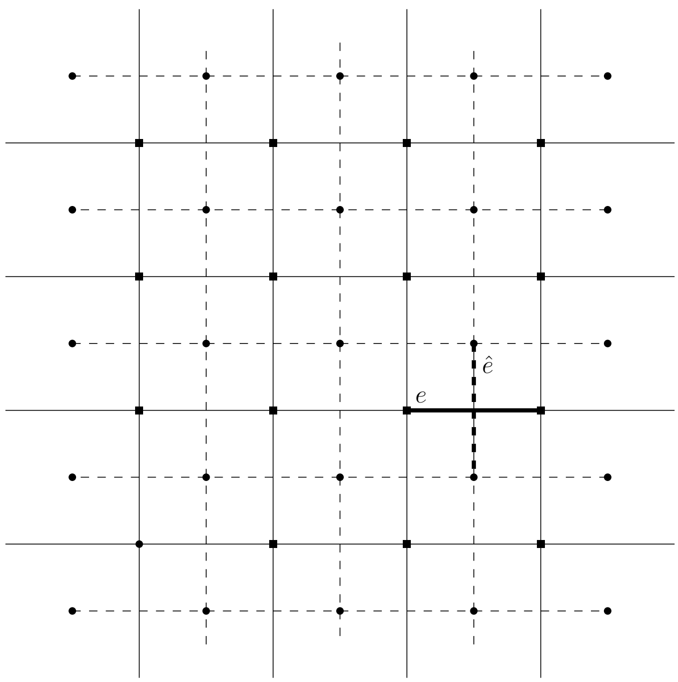


Figure 4.2: $\mathbb{Z}^2$ and it's dual.

Proposition 4.3.1 Π is a minimal cut-set in $\mathbb{Z}^2$ (with respect to 0) if and only if $\hat{\Pi} := \{\hat{e} \ : \ e \in \Pi\}$ is a self-avoiding polygon in $\widehat{\mathbb{Z}^2}$.

Proof. Let γ be a self-avoiding polygon in $\widehat{\mathbb{Z}^2}$. Then, γ splits the plane into two components, one infinite and the other finite, but non-empty. This finite component contains at least one face of $\widehat{\mathbb{Z}^2}$ so a vertex of $\mathbb{Z}^2$. So the edges dual to γ are a cut-set.

Now, let Π be a minimal cut-set in $\mathbb{Z}^2$. If F is a face in $\mathbb{Z}^2$ adjacent to an edge $e \in \Pi$, then there must be another edge $e' \neq e$ in Π that is adjacent to F; if there wasn't then one could go around the face F without crossing e, so $\Pi \setminus \{e\}$ would still be a cut-set.

So for $\gamma = \hat{\Pi}$, every vertex that γ passes through has degree at least 2. Since γ is finite, it contains a self-avoiding polygon, say γ'. However, if we go back with duality, $\Pi' := \hat{\gamma'}$ is a cut-set contained in Π, and so must be $\Pi' = \Pi$ by minimality of Π. So $\gamma' = \gamma$ and γ is a self-avoiding polygon.

Now, for the other direction, let γ be a self-avoiding polygon in $\widehat{\mathbb{Z}^2}$. So the dual of γ, $\Pi = \hat{\gamma}$ is a cut-set. Let $\Pi' \subset \Pi$ be a minimal cut-set. Then the dual $\gamma' = \hat{\Pi}'$ is a self-avoiding polygon, and also $\gamma' \subset \gamma$. But any self-avoiding polygon cannot strictly contain another self-avoiding polygon, so $\gamma' = \gamma$, and so $\Pi = \Pi'$. This implies that Π is a minimal cut-set. $\square$

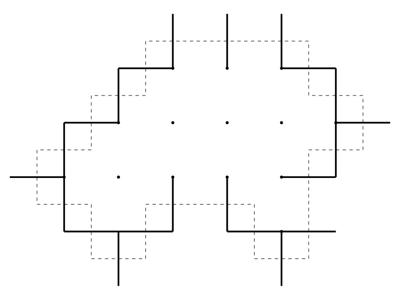


Figure 4.3: A minimal cut-set and it's dual self-avoiding polygon.

We can now use Proposition 4.3.1 to count the number of minimal cut-sets of size n in $\mathbb{Z}^2$.

Let Π be a minimal cut-set in $\mathbb{Z}^2$ of size $|\Pi| = n$. If Π does not intersect the edges of the box $\{z : ||z||_\infty \leq n\}$ then the isoperimetric inequality in $\mathbb{Z}^2$ tells us that $|\Pi| > n$, so it must be that Π contains an edge at distance at most n from 0. Let γ be the corresponding self-avoiding polygon in $\widehat{\mathbb{Z}^2}$. Then, γ is a path of length n, with a point at distance at most n from $(\frac{1}{2}, \frac{1}{2})$. There are at most $4n^2$ choices for such a point, and then since γ is self avoiding, there can be at most $4 \cdot 3^{n-1}$ possible such paths. So the number of possibilities for γ is at most $16n^2 3^{n-1}$.

Thus, for any $\varepsilon > 0$ there exists $n_0(\varepsilon)$ such that for all $n > n_0(\varepsilon)$, the number of minimal cut-sets in $\mathbb{Z}^2$ of size n is at most $16n^2 3^{n-1} < (3 + \varepsilon)^n$. So we conclude that $p_c(\mathbb{Z}^2) \leq \frac{2+\varepsilon}{3+\varepsilon}$ for any ε which gives a bound of $p_c(\mathbb{Z}^2) \leq \frac{2}{3}$.

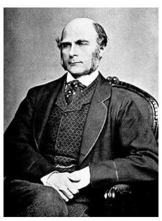
We conclude:

Corollary 4.3.2 For all $d \geq 2$,

$$\frac{1}{2d-1} \leq p_c(\mathbb{Z}^d) \leq \frac{2}{3}.$$

Chapter 5

Percolation on Trees



Francis Galton (1822-1911)

Henry Watson (1827-1903)

5.1 Galton-Watson Processes

Galton and Watson were interested in the question of the survival of aristocratic surnames in the Victorian era. They proposed a model to study the dynamics of such a family name.

In words, the model can be stated as follows. We start with one individual. This individual has a certain random number of offspring. Thus passes one generation. In the next generation, each one of the offspring has its own offspring independently. The processes continues building a random tree of descent. Let us focus only on the population size at a given generation.

Definition 5.1.1 Let μ be a distribution on $\mathbb{N}$; *i.e.* $\mu : \mathbb{N} \to [0,1]$ such that $\sum_n \mu(n) = 1$. The **Galton-Watson Process**, with offspring distribution μ, (also denoted GW_μ,) is the following Markov chain $(Z_n)_n$ on $\mathbb{N}$:

Let $(X_{j,k})_{j,k\in\mathbb{N}}$ be a sequence of i.i.d. random variables with distribution μ.

- At generation $n = 0$ we set $Z_0 = 1$. [Start with one individual.]

- Given Z_n, let
$$Z_{n+1} := \sum_{k=1}^{Z_n} X_{n+1,k}.$$
[$X_{n+1,k}$ represents the number of offspring of the k-th individual in generation n.]

Example 5.1.2 If $\mu(0) = 1$ then the GW_μ process is just the sequence $Z_0 = 1, Z_n = 0$ for all $n > 0$.

If $\mu(1) = 1$ then GW_μ is $Z_n = 1$ for all n.

How about $\mu(0) = p = 1-\mu(1)$? In this case, $Z_0 = 1$, and given that $Z_n = 1$, we have that $Z_{n+1} = 0$ with probability p, and $Z_{n+1} = 1$ with probability $1 - p$, independently of all $(Z_k \ : \ k \leq n)$. If $Z_n = 0$ the $Z_{n+1} = 0$ as well.

What is the distribution of $T = \inf\{n \ : \ Z_n = 0\}$? Well, on can easily check that $T \sim \mathrm{Geo}(p)$. So GW_μ is essentially a geometric random variable.

We will in general assume that $\mu(0) + \mu(1) < 1$, otherwise the process is not interesting. △▽△

5.2 Generating Functions

✓ Notation: For a function $f : \mathbb{R} \to \mathbb{R}$ we write $f^{(n)} = f \circ \cdots \circ f$ for the composition of f with itself n times.

Let X be a random variable with values in $\mathbb{N}$. The **probability generating function**, or PGF, is defined as
$$G_X(z) := \mathbb{E}[z^X] = \sum_n \mathbb{P}[X = n] z^n.$$

This function can be thought of as a function from $[0, 1]$ to $[0, 1]$. If $\mu(n) = \mathbb{P}[X = n]$ is the density of X, then we write $G_\mu = G_X$.

Some immediate properties:

Exercise 5.1 Let G_X be the probability generating function of a random variable X with values in $\mathbb{N}$. Show that

- If $z \in [0,1]$ then $0 \le G_X(z) \le 1$.
- $G_X(1) = 1$.
- $G_X(0) = \mathbb{P}[X = 0]$.
- $G'_X(1-) = \mathbb{E}[X]$.
- $\mathbb{E}[X^2] = G''_X(1-) + G'_X(1-)$.
- $\frac{\partial^n}{\partial z^n} G_X(0+) = n!\,\mathbb{P}[X = n]$.

◇◇◇

Proposition 5.2.1 When X takes values in $\{0\} \cup [1, \infty)$ then the PGF G_X is convex on $[0,1]$.

Proof. G_X is twice differentiable, with

$$G''_X(z) = \mathbb{E}[X(X-1)z^{X-2}] \ge 0.$$

□

The PGF is an important tool in the study of Galton-Watson processes.

Proposition 5.2.2 Let $(Z_n)_n$ be a GW_μ process. For $z \in [0,1]$,

$$\mathbb{E}[z^{Z_{n+1}} \mid Z_0, \ldots, Z_n] = G_\mu(z)^{Z_n}.$$

Thus,

$$G_{Z_n} = G_\mu^{(n)} = G_\mu \circ \cdots \circ G_\mu.$$

Proof. Conditioned on $Z_0, \ldots, Z_n$, we have that

$$Z_{n+1} = \sum_{k=1}^{Z_n} X_k,$$

where $X_1, \ldots,$ are i.i.d. distributed according to μ. Thus,

$$\mathbb{E}[z^{Z_{n+1}} \mid Z_0, \ldots, Z_n] = \mathbb{E}[\prod_{k=1}^{Z_n} z^{X_k} \mid Z_0, \ldots, Z_n] = \prod_{k=1}^{Z_n} \mathbb{E}[z^{X_k}] = G_\mu(z)^{Z_n}.$$

Taking expectations of booths sides we have that

$$G_{Z_{n+1}}(z) = \mathbb{E}[z^{Z_{n+1}}] = \mathbb{E}[G_\mu(z)^{Z_n}] = G_{Z_n}(G_\mu(z)) = G_{Z_n} \circ G_\mu(z).$$

An inductive procedure gives

$$G_{Z_n} = G_{Z_{n-1}} \circ G_\mu = G_{Z_{n-2}} \circ G_\mu \circ G_\mu = \cdots = G_\mu^{(n)},$$

since $G_{Z_1} = G_\mu$. $\square$

5.3 Extinction

Recall that the first question we would like to answer is the extinction probability for a GW process.

Let $(Z_n)_n$ be a GW_μ process. **Extinction** is the event $\{\exists n \ : \ Z_n = 0\}$. The **extinction probability** is defined to be $q = q(\mathsf{GW}_\mu) = \mathbb{P}[\exists n \ : \ Z_n = 0]$. Note that the events $\{Z_n = 0\}$ form an increasing sequence, so

$$q(\mathsf{GW}_\mu) = \lim_{n\to\infty} \mathbb{P}[Z_n = 0].$$

Proposition 5.3.1 Consider a GW_μ. (Assume that $\mu(0) + \mu(1) < 1$.) Let $q = q(\mathsf{GW}_\mu)$ be the extinction probability and $G = G_\mu$. Then,

- q is the smallest solution to the equation $G(z) = z$. If only one solution exists, $q = 1$. Otherwise, $q < 1$ and the only other solution is $G(1) = 1$.
- $q = 1$ if and only if $G'(1-) = \mathbb{E}[X] \leq 1$.

✓ Positivity of the extinction probability depends only on the mean number of offspring!

Proof. If $\mathbb{P}[X = 0] = G(0) = 0$ then $Z_n \geq Z_{n-1}$ for all n, so $q = 0$, because there is never extinction. Also, the only solutions to $G(z) = z$ in this case are $0, 1$ because $G''(z) > 0$ for $z > 0$ so G is strictly convex on $(0, 1)$, and thus $G(z) < z$ for all $z \in (0, 1)$. So we can assume that $G(0) > 0$.

Let $f(z) = G(z) - z$. So $f''(z) > 0$ for $z > 0$. Thus, f' is a strictly increasing function.

- Case 1: If $G'(1-) \leq 1$. So $f'(1-) \leq 0$. Since $f'(0+) = -(1 - \mu(1)) < 0$ (because $\mu(1) < 1$), and since f' is strictly increasing, for all $z < 1$ we have that $f'(z) < 0$. Thus, the minimal value of f is at 1; that is, $f(z) > 0$ for all $z < 1$ and there is only one solution to $f(z) = 0$ at 1.

- Case 2: If $G'(1-) > 1$. Then $f'(1-) > 0$. Since $f'(0+) < 0$ there must be some $0 < x < 1$ such that $f'(x) = 0$. Since f' is strictly increasing, this is the unique minimum of f in $[0, 1]$. Since $f'(z) > 0$ for $z > x$, as a minimum, we have that $f(x) < f(\frac{1+x}{2}) \leq f(1) = 0$. Also, $f(0) = \mu(0) > 0$, and because f is continuous, there exists a $0 < p < x$ such that $f(p) = 0$.

  We claim that $p, 1$ are the only solutions to $f(z) = 0$. Indeed, if $0 < a < b < 1$ are any such solutions, then because f is strictly convex on $(0, 1)$, since we can write $b = \alpha a + (1 - \alpha)1$ for some $\alpha \in (0, 1)$, we have that $0 = f(b) < \alpha f(a) + (1 - \alpha) f(1) = 0$, a contradiction.

  In conclusion, in the case $G'(1-) > 1$ we have that there are exactly two solutions to $G(z) = z$, which are p and 1.

  Moreover, $p < x$ for x the unique minimum of f, so because f' is strictly increasing,

  $$-1 \leq -(1 - \mu(1)) = f'(0+) \leq f'(z) \leq f'(p) < f'(x) = 0$$

  for any $z \leq p$. Thus, for any $z \leq p$ we have that

  $$f(z) = f(z) - f(p) = -\int_z^p f'(t)dt \leq p - z,$$

  which implies that $G(z) \leq p$ for any $z \leq p$.

Now, recall that the extinction probability admits

$$q = \lim_{n \to \infty} \mathbb{P}[Z_n = 0] = \lim_{n \to \infty} G_{Z_n}(0) = \lim_{n \to \infty} G^{(n)}(0).$$

Since G is a continuous function, we get that $G(q) = q$ so q is a solution to $G(z) = z$.

If two solutions exists (equivalently, $G'(1-) > 1$), say p and 1, then $G^{(n)}(0) \leq p$ for all n, so $q \leq p$ and thus must be $q = p < 1$.

If only one solution exists then $q = 1$. $\square$

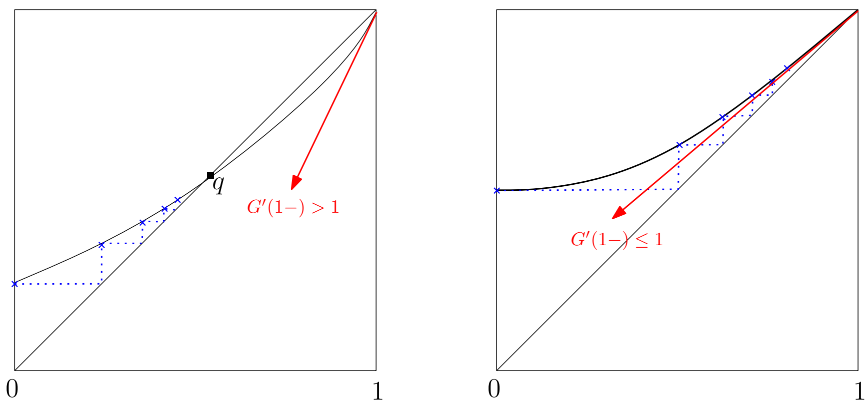


Figure 5.1: The two possibilities for $G'(1-)$. The blue dotted line and crosses show how the iterates $G^{(n)}(0)$ advance toward the minimal solution of $G(z) = z$.

5.4 Percolation on Regular Trees

Let $d \geq 3$ and let T be a rooted d-regular tree; that is, a tree such that every vertex has degree d except for the root which has degree $d - 1$. Let $o \in T$ be the root vertex. Note that every vertex that is not the root has a unique ancestor, which is the neighbor that is closer to the root. Also, every vertex has $d - 1$ neighbors farther from the root, which we call descendants.

For each n let $T_n = \{x : \text{dist}(x, o) = n\}$. If we perform p-percolation on T, which vertices are in $\mathcal{C}(o)$?

Well, for a vertex $x \neq o$ let y be the unique ancestor of x; that is let y be the unique neighbor of x that is closer to o. Then, $x \leftrightarrow o$ if and only if the edge $\{y, x\}$ is open and $y \leftrightarrow o$.

Now, let Z_n be the number of vertices in T_n that are in $\mathcal{C}(o)$; $Z_n = |\mathcal{C}(o) \cap T_n|$. Then, we have that

$$Z_n = \sum_{y \in \mathcal{C}(o) \cap T_{n-1}} \sum_{x \in T_n, x \sim y} \mathbf{1}_{\{\Omega_p(\{y,x\})=1\}}.$$

Since all these edges are independent, for each $y \in \mathcal{C}(o) \cap T_{n-1}$, we have that $\sum_{x \in T_n, x \sim y} \mathbf{1}_{\{\Omega_p(\{y,x\})=1\}} \sim \mathrm{Bin}(d-1, p)$. Also from independence we get that given Z_{n-1},

$$Z_n = \sum_{j=1}^{Z_{n-1}} B_j,$$

where $(B_j)_j$ are independent $\mathrm{Bin}(d-1, p)$ random variables.

So $(Z_n)_n$ constitute a Galton-Watson process with offspring distribution $\mathrm{Bin}(d-1, p)$.

We thus conclude:

Proposition 5.4.1 If T is a rooted d-regular tree, with root $o \in T$, then the sequence $(|\mathcal{C}(o) \cap T_n|)_n$ is a Galton-Watson process with offspring distribution $\mathrm{Bin}(d-1, p)$. (Here T_n is the n-th level of the tree T.)

Theorem 5.4.2 For $d \geq 3$, and percolation on the d-regular tree $\mathbb{T}_d$:

- $p_c(\mathbb{T}_d) = \frac{1}{d-1}$.
- $\Theta_{\mathbb{T}_d}(p_c) = 0$.

Proof. Let $o \in \mathbb{T}_d$ be some vertex. Let $x_1, \ldots, x_d$ be the d neighbors of o. For every $y \neq o$ there is a unique j such that the shortest path from y to o must go through x_j. We call such a y a descendant of x_j (with respect to o). Let T_j be the subtree of descendants of x_j.

Let $T = T_1$ which is a rooted d-regular tree rooted at x_1. Since T is a rooted d-regular tree, the component of x_1 in p-percolation on T is infinite if and only if a $\mathrm{Bin}(d-1, p)$ Galton-Watson process survives. That is, $\theta_{T,x_1}(p) > 0$ if and only if $p(d-1) > 1$. So $p_c(T) \leq \frac{1}{d-1}$.

T is a subgraph of $\mathbb{T}_d$, which implies that $p_c(\mathbb{T}_d) \leq p_c(T) \leq \frac{1}{d-1}$. Also, the (maximal) degree in $\mathbb{T}_d$ is d so $p_c(\mathbb{T}_d) \geq \frac{1}{d-1}$.

So $p_c(\mathbb{T}_d) = p_c(T) = \frac{1}{d-1}$, which is the first assertion.

For the second assertion, note that $o \leftrightarrow \infty$ if and only if there exists j such that $|\mathcal{C}(x_j) \cap T_j| = \infty$. Thus,

$$\mathbb{P}_p[o \leftrightarrow \infty] \leq \sum_{j=1}^{d} \mathbb{P}_p[|\mathcal{C}(x_j) \cap T_j| = \infty] = d \cdot \theta_{T_j,x_j}(p).$$

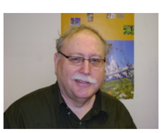
For $p = \frac{1}{d-1}$, the Galton-Watson process above does not survive a.s. (because $p(d-1) = 1$). So $\theta_{T_j,x_j}(\frac{1}{d-1}) = 0$, and so $\theta_{\mathbb{T}_d,o}(\frac{1}{d-1}) = 0$. $\square$

Chapter 6

The Number of Infinite Components

Michael Aizenman

Harry Kesten



Charles Newman

6.1 The Number of Infinite Clusters

In 1987 Aizenman, Kesten and Newman proved that for percolation on $\mathbb{Z}^d$ the infinite component is unique if it exists. Two years later, Burton and Keane provided a short proof of this result that works for all amenable transitive graphs.

Lemma 6.1.1 Let G be a transitive infinite connected graph. For percolation on G let N be the number of infinite components. Then, for any $p \in (0,1)$ there exists $k \in \{0, 1, \infty\}$ such that $\mathbb{P}_p[N = k] = 1$.

✓ Notation: We require some more notation regarding random variables $X : \{0,1\}^{E(G)} \to \mathbb{R}$. Let $\eta \in \{0,1\}^{E(G)}$ be some vector, and let $E \subset E(G)$. For $\omega \in \{0,1\}^{E(G)}$ define $\omega_{\eta,E}(e) = \eta(e)\mathbf{1}_{\{e\in E\}} + \omega(e)\mathbf{1}_{\{e\notin E\}}$; that is, the values of ω on coordinates in E are changed to match those in η. (Note for example that the cylinder around E at η is just $C_{\eta,E} = \{\omega \ : \ \omega = \omega_\eta\}$.) Specifically, for the case that $\eta \equiv 1$ of $\eta \equiv 0$ we write

$$\omega_{1,E}(e) = \mathbf{1}_{\{e\in E\}} + \mathbf{1}_{\{e\notin E\}}\omega(e) \qquad \text{and} \qquad \omega_{0,E}(e) = \mathbf{1}_{\{e\notin E\}}\omega(e).$$

$\omega_{1,E}$ (resp. $\omega_{0,E}$) is the configuration ω after forcing all edges in E to be open (resp. closed).

For a random variable $X : \{0,1\}^{E(G)} \to \mathbb{R}$ define $X_{\eta,E}(\omega) = X(\omega_{\eta,E})$. That is, $X_{\eta,E}$ measures X when the configuration in E is forced to be η. As above, let $X_{1,E}(\omega) = X(\omega_{1,E})$ and $X_{0,E}(\omega) = X(\omega_{0,E})$.

Exercise 6.1 Let G be a graph, and let $\eta \in \{0,1\}^{E(G)}$ and $E \subset E(G)$. Let $X : \{0,1\}^{E(G)} \to \mathbb{R}$ be a random variable. Show that $X_{\eta,E}$ is measurable with respect to $\mathcal{T}_E$ (that is, $X_{\eta,E}$ does not depend on the edges in E). ◇◇◇

Proof of Lemma 6.1.1. Recall Lemma 2.2.1 that states that any translation invariant event has probability either 0 or 1. For any $k \in \{0,1,\ldots,\infty\}$ the event $\{N = k\}$ is translation invariant. Thus, $\mathbb{P}_p[N = k] \in \{0,1\}$. Thus, there is a unique $k = k_p \in \{0,1,\ldots,\infty\}$ such that $\mathbb{P}_p[N = k] = 1$.

Let B be some finite subset of G. Let $E(B)$ be the set of edges with both endpoints in B. We assume that B is such that $E(B) \neq \emptyset$. Let C_B be the event that all edges in $E(B)$ are closed, and let O_B be the event that all edges in $E(B)$ are open. As long as $p \in (0,1)$ and $E(B) \neq \emptyset$, these events have positive probability.

Now, let N_C be the number of infinite components if we declare all edges in $E(B)$ to be closed, and let N_O be the number of infinite components if we declare all edges in $E(B)$ to be open. That is, $N_C = N_{0,E(B)}$ and $N_O = N_{1,E(B)}$. Note that $\omega_C = \omega$ if and only if $\omega \in C_B$ and $\omega_O = \omega$ if and only if $\omega \in O_B$. Also, note that $N_C, N_O \in \mathcal{T}_{E(B)}$ and thus are independent of the edges in $E(B)$. So, using the fact that C_B, O_B have positive probability, for $k = k_p$,

$$\mathbb{P}_p[N_C = k] = \mathbb{P}_p[N_C = k | C_B] = \mathbb{P}_p[N_C = N = k | C_B] = \mathbb{P}_p[N = k | C_B] = 1,$$

and similarly,

$$\mathbb{P}_p[N_O = k] = \mathbb{P}_p[N_O = k | O_B] = \mathbb{P}_p[N_O = N = k | O_B] = 1.$$

So $\mathbb{P}_p$-a.s. $N_C = N_O = k$.

Let N_B be the number of infinite components that intersect B. Opening all edges in $E(B)$ connects all components intersecting B, and closing all edges in $E(B)$ disconnects them. So if $N_B \geq 2$ and $N < \infty$, then $N_O \leq N - 1$ and $N_C \geq N$.

Thus, we conclude that for all $p \in (0,1)$, if $k_p < \infty$, we have that

$$\mathbb{P}_p[N_B \geq 2] \leq \mathbb{P}_p[N_O \leq N_C - 1] = 0.$$

Fix some $o \in G$. For every r use B above as the ball of radius r around o. Let N_r be the number of infinite components intersecting the ball of radius r around o. If $k_p < \infty$ we get that $\mathbb{P}_p[N_r \geq 2] = 0$. Since $N_r \nearrow N$, we conclude that if $k_p < \infty$, $\mathbb{P}_p[N \geq 2] = 0$. So $k_p \in \{0, 1, \infty\}$ as claimed. $\square$

Exercise 6.2 Give an example of a transitive graph and some $p \in (0,1)$ for which the number of infinite components in percolation is a.s. ∞.
◇ ◇ ◇

John von Neumann (1903–1957)

6.2 Amenable Graphs

Definition 6.2.1 Let G be a connected graph. Let $S \subset G$ be a finite subset. Define the (outer) **boundary** of S to be

$$\partial S = \{x \notin S \ : \ x \sim S\}.$$

Define the **isoperimetric constant** of G to be

$$\Phi = \Phi(G) := \inf\{|\partial S|/|S| \ : \ S \text{ is a finite connected subset of } G\}.$$

Of course $D \geq \Phi(G) \geq 0$ for any graph, where D is the maximal degree in G. When $\Phi(G) > 0$, we have that sets "expand": the boundary of any set is proportional to the volume of the set.

Definition 6.2.2 Let G be a graph. If $\Phi(G) = 0$ we say that G is **amenable**. Otherwise we call G **non-amenable**.

A sequence of finite connected sets $(S_n)_n$ such that $|\partial S_n|/|S_n| \to 0$ is called a Folner sequence, and the sets are called Folner sets.

Erling Folner (1919–1991)

The concept of amenability was introduced by von Neumann in the context of groups and the Banach-Tarski paradox. Folner's criterion using boundaries of sets provided the ability to carry over the concept of amenability to other geometric objects such as graphs.

Exercise 6.3 Let $S \subset \mathbb{T}_d$ be a finite connected subset, with $|S| \geq 2$. Show that $|\partial S| = |S|(d-2) + 2$.

Deduce that $\Phi(\mathbb{T}_d) = d - 2$. ⋄⋄⋄

Exercise 6.4 Show that $\mathbb{Z}^d$ is amenable. ⋄⋄⋄

6.3 The Burton-Keane Theorem

Burton and Keane's argument consist of the definition of a trifurcation point: a point where 3 different clusters meet.

Definition 6.3.1 Let G be an infinite connected graph. Let $x \in G$. $B(x,r)$ denotes the ball of radius r around x (in the graph metric). For percolation on G, let N_r be the number of infinite clusters that intersect $B(x,r)$. Let M_r be the number of infinite clusters that intersect $B(x,r)$ if we declare all edges in $B(x,r)$ closed; *i.e.* $M_r = (N_r)_{0,E(B(x,r))}$. Let $\Psi_r(x)$ be the event that x is a r-**trifurcation point**, defined as follows:

- $B(x,r)$ intersects an infinite cluster ($N_r \geq 1$).
- If we closed all edges in $B(x,r)$ then the number of infinite clusters that the $B(x,r)$ intersect split into at least 3 infinite clusters ($M_r \geq 3$).

Let us sketch a proof of why there cannot be infinitely many infinite clusters in transitive amenable graphs. If there are infinitely many infinite clusters, then three different ones should meet at some ball of radius r, so that the center of that ball is a r-trifurcation point. That is, there is positive probability for any point to be a trifurcation point. Transitivity gives that we expect to see a positive proportion of trifurcation points in any finite set. But the number of trifurcation points cannot be more than the number of boundary points (by some combinatorial considerations). If the boundary is much smaller than the volume of the set, we obtain a contradiction.

The rigorous argument relies on two main lemmas:

Robert Burton

Lemma 6.3.2 Let G be a transitive infinite connected graph. Let N be the number of infinite clusters in percolation on G.

For any $p \in (0,1)$, there exists $r > 0$ such that if $N = \infty$ $\mathbb{P}_p$ −a.s., then for any x, $\mathbb{P}_p[\Psi_r(x)] > 0$.

Proof. For every $r > 0$ let N_r be the number of infinite clusters that intersect $B(x,r)$. Let M_r be the number of infinite clusters that intersect $B(x,r)$ if we declare all edges in $B(x,r)$ closed.

If $N = \infty$ $\mathbb{P}_p$-a.s. then since $N_r \nearrow N$ we have that for some $r = r(p) > 0$, $\mathbb{P}_p[N_r \geq 3] \geq \frac{1}{2}$.

The main observation here, is that if we close the edges in $B(x,r)$. then any infinite component that intersects $B(x,r)$ splits into components that intersect $B(x,r)$, one of which must be infinite. That is, $M_r \geq N_r$ a.s. So we have that

$$\mathbb{P}_p[\Psi_r(x)] \geq \mathbb{P}_p[M_r \geq 3, N_r \geq 1] \geq \mathbb{P}_p[N_r \geq 3] \geq \frac{1}{2}.$$

□

Michael Keane

The next lemma is a bit more combinatorial, and will be proven in the next section.

Lemma 6.3.3 Let G be an infinite connected graph. Let S be a finite connected subset of G. For percolation on G, the number of r-trifurcation points in S is at most $D^r|\partial S|$, where D is the maximal degree in G.

These Lemmas lead to the following

Theorem 6.3.4 Let G be a transitive infinite connected graph. Assume that G is amenable. Then, for any $p \in (0,1)$ there exists $k = k_p \in \{0,1\}$ such that the number of infinite clusters in p-percolation on G is $\mathbb{P}_p$ −a.s. k.

In other words, there is either a.s. no infinite cluster or a.s. a unique infinite cluster.

Proof. Let $o \in G$ be some fixed vertex. Let $(S_n)_n$ be a sequence of Folner sets. Let $r = r(p) > 0$ be such that $\mathbb{P}_p[\Psi_r(o)] > 0$. For each n, let τ_n be the number of r-trifurcation points in S_n. Note that by transitivity, using Lemma 6.3.3,

$$d^r \cdot |\partial S_n| \geq \mathbb{E}_p[\tau_n] = \sum_{s \in S_n} \mathbb{P}_p[\Psi_r(s)] = |S_n| \cdot \mathbb{P}_p[\Psi_r(o)],$$

where d is the degree in G. So,

$$\mathbb{P}_p[\Psi_r(o)] \leq d^r \cdot \frac{|\partial S_n|}{|S_n|} \to 0.$$

By Lemma 6.3.2 this implies that the number of infinite clusters is not ∞ $\mathbb{P}_p$ −a.s. So this number is either 0 or 1 by the $0, 1, \infty$ law. □

6.4 Proof of Lemma 6.3.3

Proof. Assume first that for every $s \in S$, also $B(s, r) \subset S$.

Define an auxiliary graph Ψ: The vertices of Ψ are the r-trifurcation points in S and the points in ∂S that are in infinite components. For edges of Ψ, let $x \sim y$ if $B(x, r) \leftrightarrow B(y, r)$ by an open path that does not pass through any $B(z, r)$ for some other vertex z, and *lies on some infinite component.*

Note that by definition, if x is a r-trifurcation point, then in the graph Ψ, removing x will split Ψ into at least 3 connected components. (This is not necessarily true for vertices of Ψ which are not trifurcation points.) Here is where we use the assumption on S. Indeed, if x is a r-trifurcation point then there are three infinite clusters that intersect $S \setminus B(x, r)$, that cannot connect outside of $B(x, r)$. That is, there are at least three points in ∂S that are vertices of Ψ. These points cannot connect to one another in the graph Ψ: Indeed, if some part of an infinite path connects two balls around such points, say $B(a, r), B(b, r)$, then this infinite path must intersect another boundary point, say c. If $c \neq a, b$ then the infinite path goes through $B(c, r)$ so would not connect a, b in the graph Ψ. If $c = a$ (or $c = b$) then the infinite path would connect the two components coming from $B(x, r)$, so x would not be a trifurcation point.

We claim:

Claim Let Ψ be a graph and let $V \subset \Psi$ be a set of vertices with the property that for every vertex $v \in V$, the graph induced on $\Psi \setminus \{v\}$ has at least 3 connected components. Then $2|V| + 2 \leq |\Psi|$.

Proof of Claim. By induction on $|V|$. If $|V| = 1$ then this is obvious, since $|\Psi \setminus \{v\}| \geq 3$ for $V = \{v\}$.

Now, if $|V| = n+1$, then let $v \in V$ and let $C_1, \ldots, C_k$ be the components of $\Psi \setminus \{v\}$. For every $j = 1, 2$, let w_j be an auxiliary vertex, and consider the graph Ψ_j where w_j is connected to C_j instead of v. For $j = 3$ let w_3 be an auxiliary vertex, and let Ψ_3 be the graph where w_3 is connected to $C_3, \ldots, C_k$ instead of v.

Let $V_j = V \cap C_j$. Then, $|V_j| \leq n$, and also, for every $v \in V_j$, if we remove v from Ψ_j then there are at least 3 components in $\Psi_j \setminus \{v\}$ (otherwise $\Psi \setminus \{v\}$ would not have at least 3 components). By induction, if $V_j \neq \emptyset$ then $2|V_j| + 2 \leq |\Psi_j|$ for all $j = 1, 2, 3$ and if $V_j = \emptyset$ then this holds because $|\Psi_j| \geq 2$. So,

$$2|V| = 2+2(|V_1|+|V_2|+|V_3|) \leq 2+|\Psi_1|+|\Psi_2|+|\Psi_3|-6 = |\Psi|+2-4 = |\Psi|-2.$$

□

Using the claim, we get that if A is the number of r-trifurcation points in S, then $2A \leq |\Psi| - 2 = A + (|\Psi| - A) - 2$. Since any vertex of Ψ that is not a trifurcation point is a boundary point, we conclude $2A \leq A + |\partial S| - 2$ so $A < |\partial S|$.

All this was under the assumption that for every $s \in S$, also $B(s,r) \subset S$. For general S, let $S' = \bigcup_{s \in S} B(s,r)$. Then any r-trifurcation point in S is a r-trifurcation point in S'. So it suffices to show that $|\partial S'| \leq D^r |\partial S|$.

Now, if D is the maximal degree in G, then since for any x, $|\partial B(x,r)| \leq D(D-1)^r$, and since any point in $\partial S'$ is at distance exactly r from a point in ∂S, we have that

$$|\partial S'| \leq \sum_{x \in \partial S} |\partial B(x, r-1)| \leq D^r \cdot |\partial S|.$$

□

Chapter 7

Probabilistic Tools for Product Spaces

7.1 Disjointly Occurring Events

Suppose x, y, z are three vertices in a graph G. We can ask about the events $x \leftrightarrow y, x \leftrightarrow z$ in percolation on G. This is equivalent to the event that there exist open paths $\alpha : x \to y$ and $\beta : x \to z$. We can also consider a smaller event: the event that there exists open paths $\alpha : x \to y$ and $\beta : x \to z$ *that are disjoint*. This is the canonical example of the two events $x \leftrightarrow y$ and $x \leftrightarrow z$ occurring *disjointly*. That is, the events are required to both occur, but each must be guarantied by a disjoint set of edges.

Let us give a formal definition. Recall that for a set of edges $E \subset E(G)$ and a vector $\eta \in \{0,1\}^{E(G)}$, the cylinder of E and η is

$$C_{\eta,E} = \{\omega \ : \ \forall e \in E \ , \ \omega(e) = \eta(e)\} .$$

For an event A, a set of edges E, and a configuration $\omega \in \{0,1\}^{E(G)}$, we say that E **guaranties** A at ω if $C_{\omega,E} \subset A$. In words, knowing that on the edges in E the configuration is ω implies the event A.

Definition 7.1.1 For two events A, B define an event

$$A \circ B := \left\{\omega \ : \ \exists \ E \cap E' = \emptyset \ , \ C_{\omega,E} \subset A, C_{\omega,E'} \subset B\right\} .$$

That is, $A \circ B$ is the event that both A, B occur, but their occurrence is guarantied by disjoint edge sets.

Exercise 7.1 Show that $A \circ B \subset A \cap B$. ◇◇◇

Exercise 7.2 Show that if A is increasing and B is decreasing then $A \circ B = A \cap B$. ◇◇◇

Exercise 7.3 Show that if $A \subset A', B \subset B'$ then $A \circ B \subset A' \circ B'$. ◇◇◇

7.2 The BK Inequality

Rob van den Berg

The following is a useful inequality proved by van den Berg and Kesten. It has come to be known as the BK inequality.

Theorem 7.2.1 (BK Inequality) Consider percolation on G. Let $A, B \in \mathcal{F}_E$ be increasing events depending on some finite set of edges $E \subset E(G), |E| < \infty$. Then,

$$\mathbb{P}_p[A \circ B] \leq \mathbb{P}_p[A] \cdot \mathbb{P}_p[B].$$

Before proving the BK inequality, let us remark that if A was increasing and B decreasing, then the inequality holds by Harris' Lemma. Moreover, with a much more complicated proof, Reimer proved that the assumption of increasing is unnecessary:

Theorem 7.2.2 (Reimer's Inequality) Consider percolation on G. Let $A, B \in \mathcal{F}_E$ be any events depending on some finite set of edges $E \subset E(G), |E| < \infty$. Then,

$$\mathbb{P}_p[A \circ B] \leq \mathbb{P}_p[A] \cdot \mathbb{P}_p[B].$$

The main observation that makes proving BK easier than proving Reimer's inequality is

Claim 7.2.3 If A, B are both increasing events, then $A \circ B$ has a simpler form:

$$A \circ B = \{\omega + \eta \ : \ \omega \cdot \eta = 0 \ , \ \omega \in A, \eta \in B\} .$$

Proof. We show two inclusions.

For "$\supset$": Let $\omega \cdot \eta = 0$ and $\omega \in A, \eta \in B$. Let $E = \omega^{-1}(1), E' = \eta^{-1}(1)$. Then $\omega\eta \equiv 0$ implies $E \cap E' = \emptyset$. Because A, B are increasing,

$$C_{\omega,E} \subset A \qquad \text{and} \qquad C_{\eta,E'} \subset B.$$

Also, since $C_{\omega+\eta,E} = C_{\omega,E}$ and $C_{\omega+\eta,E'} = C_{\eta,E'}$,

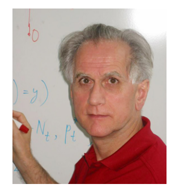
$$C_{\omega+\eta,E} \subset A \qquad \text{and} \qquad C_{\omega+\eta,E'} \subset B.$$

So $\omega + \eta \in A \circ B$.

For "$\subset$": Let $\xi \in A \circ B$. Let $E \cap E' = \emptyset$ be such that $C_{\xi,E} \subset A$ and $C_{\xi,E'} \subset B$. Let

$$\omega(e) = \mathbf{1}_{\{e \in E\}}\xi(e) + \mathbf{1}_{\{e \notin E \cup E'\}}\xi(e) \qquad \text{and} \qquad \eta(e) = \mathbf{1}_{\{e \in E'\}}\xi(e).$$

So by definition $\omega + \eta = \xi$ and $\omega \cdot \eta = 0$. Note that $\omega \in C_{\xi,E} \subset A$ and $\eta \in C_{\xi,E'} \subset B$. □



Béla Bollobás

Exercise 7.4 Show that if A, B are increasing, then $A \circ B$ is increasing.
◇ ◇ ◇

Proof of the BK inequality (Bollobás & Leader). The proof is by induction on the size of the edge set $E \subset E(G)$ for which $A, B \in \mathcal{F}_E$.

The case where $|E| = 1$ is simple, because if $A, B \in \mathcal{F}_{\{e\}}$ are increasing non-trivial events, then $A = B = \mathbf{1}_{\{e \text{ is open }\}}$. Thus, $A \circ B = \emptyset$.

Suppose that $|E| > 1$. Remove an edge e from E, and let $E' = E \setminus \{e\}$. For any event $F \in \mathcal{F}_E$ and $j \in \{0, 1\}$ let $F_j \in \mathcal{F}_{E'}$ be defined by

$$F_j = \{\omega \ : \ \omega_{j,e} \in F\}.$$

(Recall that $\omega_{j,e}(e') = \mathbf{1}_{\{e'=e\}}j + \mathbf{1}_{\{e' \neq e\}}\omega(e')$.)

Let $D = A \circ B$.

Claim 1. A_j, B_j are increasing for $j \in \{0, 1\}$. Indeed, if $A_j \ni \omega \leq \eta$, then $\omega_{j,e} \in A$. We always have $\eta_{j,e} \geq \omega_{j,e}$. Since A is increasing, $\eta_{j,e} \in A$, which gives $\eta \in A_j$.

Claim 2. $D_0 \subset A_0 \circ B_0$. (Actually, equality holds, but this will suffice.)

If $\xi \in D_0$ then $\xi_{0,e} = \omega + \eta$ for $\omega \cdot \eta = 0$ and $\omega \in A, \eta \in B$. So $\omega(e) = \eta(e) = 0$, and $\omega \in A_0, \eta \in B_0$. So $\xi_{0,e} \in A_0 \circ B_0$. But $A_0 \circ B_0$ is increasing, and $\xi \geq \xi_{0,e}$, so $\xi \in A_0 \circ B_0$.

Claim 3. $D_1 \subset (A_0 \circ B_1) \cup (A_1 \circ B_0)$. (Actually, equality holds, but this will suffice.)

If $\xi \in D_1$, then $\xi_{1,e} = \omega + \eta$ for $\omega \cdot \eta = 0$ and $\omega \in A, \eta \in B$. Thus, either $\omega(e) = 1 = 1 - \eta(e)$ or $\eta(e) = 1 = 1 - \omega(e)$. In the first case, $\omega_{1,e} = \omega \in A, \eta_{0,e} = \eta \in B$ and in the second case $\omega_{0,e} \in A, \eta_{1,e} \in B$. So either $\omega \in A_1, \eta \in B_0$ or $\omega \in A_0, \eta \in B_1$. Thus, $\omega + \eta \in (A_0 \circ B_1) \cup (A_1 \circ B_0)$.

Imre Leader

Now, if $\xi(e) = 1$ then $\xi = \xi_{1,e} = \omega + \eta$ and we are done. If $\xi(e) = 0$ then $\xi = \omega_{0,e} + \eta_{0,e}$ and $\omega_{0,e} \cdot \eta_{0,e} = 0$. So

$$\xi \in A_0 \circ B_0 \subset A_1 \circ B_0 \cap A_0 \circ B_1.$$

With these claims, we continue: We have that $A_0 \subset A_1, B_0 \subset B_1$. By Claim 2 and by induction,

$$\mathbb{P}[D_0] \leq \mathbb{P}[A_0] \cdot \mathbb{P}[B_0]. \tag{7.1}$$

By Claim 3, Claim 1 and induction

$$\mathbb{P}[D_1] \leq \mathbb{P}[A_1 \circ B_1] \leq \mathbb{P}[A_1] \cdot \mathbb{P}[B_1]. \tag{7.2}$$

Claim 2 tells us that

$$D_0 \subset (A_0 \circ B_1) \cap (A_1 \circ B_0),$$

so with Claim 3,

$$\begin{aligned} \mathbb{P}[D_0] + \mathbb{P}[D_1] &\leq \mathbb{P}[(A_0 \circ B_1) \cap (A_1 \circ B_0)] + \mathbb{P}[(A_0 \circ B_1) \cup (A_1 \circ B_0)] \\ &= \mathbb{P}[A_0 \circ B_1] + \mathbb{P}[A_1 \circ B_0] \leq \mathbb{P}[A_0] \cdot \mathbb{P}[B_1] + \mathbb{P}[A_1] \cdot \mathbb{P}[B_0]. \end{aligned} \tag{7.3}$$

Finally,

$$\begin{aligned} \mathbb{P}[A] \cdot \mathbb{P}[B] &= ((1-p)\,\mathbb{P}[A_0] + p\,\mathbb{P}[A_1]) \cdot ((1-p)\,\mathbb{P}[B_0] + p\,\mathbb{P}[B_1]) \\ &= (1-p)^2 \cdot \mathbb{P}[A_0] \cdot \mathbb{P}[B_0] + (1-p)p \cdot (\mathbb{P}[A_0] \cdot \mathbb{P}[B_1] + \mathbb{P}[A_1] \cdot \mathbb{P}[B_0]) + p^2 \cdot \mathbb{P}[A_1] \cdot \mathbb{P}[B_1] \\ &\geq (1-p)^2\,\mathbb{P}[D_0] + (1-p)p\,(\mathbb{P}[D_0] + \mathbb{P}[D_1]) + p^2\,\mathbb{P}[D_1] \\ &= (1-p)\,\mathbb{P}[D_0] + p\,\mathbb{P}[D_1] = \mathbb{P}[D]. \end{aligned}$$

□

7.3 Russo's Formula

Grigory Margulis

Another useful tool is a formula discovered by Margulis and independently by Russo. It is classically dubbed *Russo's formula* in the literature.

Suppose $X : \{0,1\}^{E(G)} \to \mathbb{R}$ is some random variable. Recall the definition $\omega_{j,e}(e') = \mathbf{1}_{\{e' \neq e\}} \omega(e') + \mathbf{1}_{\{e'=e\}} j$ for $j \in \{0,1\}$. Define the *derivative* of X at e, denoted $\partial_e X$ to be the random variable

$$\partial_e X(\omega) = X(\omega_{1,e}) - X(\omega_{0,e}) = (X_{1,e} - X_{0,e})(\omega).$$

The expectation $\mathbb{E}[\partial_e X]$ is called the *influence of e on X*.

Exercise 7.5 Let A be an event. We say that an edge e is *pivotal in ω* for A if for $\omega \in A, \omega + \delta_e \notin A$ or $\omega \notin A, \omega + \delta_e \in A$. Here

$$(\omega + \delta_e)(e') = \mathbf{1}_{\{e' \neq e\}} \omega(e') + \mathbf{1}_{\{e'=e\}} (1 - \omega(e))$$

flips the value of ω at e.

That is, e is pivotal in ω for A if flipping the state of the edge e changes whether ω is in A.

- Show that

  $$\{\omega \ : \ e \text{ is pivotal in } \omega \text{ for } A \ \} = A_{1,e} \triangle A_{0,e},$$

  where $A_{j,e} = \{\omega \ : \ \omega_{j,e} \in A\}$.

- Show that if A is increasing

  $$\mathbb{E}[\partial_e \mathbf{1}_A] = \mathbb{P}[\{\omega \ : \ e \text{ is pivotal in } \omega \text{ for } A \ \}].$$

◇◇◇

Theorem 7.3.1 (Russo's Formula) For any random variable that depends

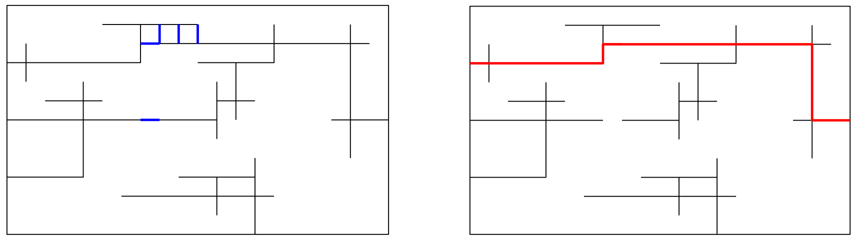


Figure 7.1: Two examples of configurations ω with edges that are pivotal in ω for the event of crossing the rectangle from left to right. On the left, blue edges are closed, and are all pivotal. On the right, red edges are all open, and are all pivotal.

on finitely many edges, $X \in \mathcal{F}_E, |E| < \infty$,

$$\frac{d}{dp}\mathbb{E}_p[X] = \sum_{e \in E} \mathbb{E}_p[\partial_e X].$$

Lucio Russo

Proof. We will in fact prove something stronger. Let $E = \{e_1, \ldots, e_n\}$. For any vector of probabilities $\mathbf{p} = (p_1, \ldots, p_n) \in [0,1]^n$ consider the probability measure $\mathbb{P}_\mathbf{p}$ (with expectation $\mathbb{E}_\mathbf{p}$) for which the edge states $\omega(e_j)$ are all independent Bernoulli random variables, with mean p_j for the edge e_j. Define a function $f : [0,1]^n \to \mathbb{R}$ by

$$f(\mathbf{p}) = \mathbb{E}_\mathbf{p}[X].$$

We will show that

$$\frac{\partial}{\partial x_j} f(\mathbf{p}) = \mathbb{E}_\mathbf{p}[\partial_{e_j} X].$$

Russo's formula above then follows by the chain rule, and taking $p_1 = \cdots = p_n = p$.

Let $U_1, \ldots, U_n$ be independent uniform-$[0,1]$ random variables, and let $\Omega_\mathbf{p} : E \to \{0,1\}$ be defined by

$$\Omega_\mathbf{p}(e_j) = \mathbf{1}_{\{U_j \le p_j\}}.$$

So $\Omega_\mathbf{p}$ has the law of $\mathbb{P}_\mathbf{p}$. With this coupling, for any j, and small ε,

$$f(\mathbf{p} + \delta_j \varepsilon) - f(\mathbf{p}) = \mathbb{E}[X(\Omega_{\mathbf{p}+\delta_j\varepsilon}) - X(\Omega_\mathbf{p})].$$

If $U_j \notin (p_j, p_j + \varepsilon]$ the $\Omega_{\mathbf{p}+\delta_j\varepsilon} = \Omega_{\mathbf{p}}$. Also, if $U_j \in (p_j, p_j + \varepsilon]$ then

$$X(\Omega_{\mathbf{p}+\delta_j\varepsilon}) = X_{1,e_j}(\Omega_{\mathbf{p}}) \qquad \text{and} \qquad X(\Omega_{\mathbf{p}}) = X_{0,e_j}(\Omega_{\mathbf{p}}).$$

So by independence,

$$f(\mathbf{p} + \delta_j\varepsilon) - f(\mathbf{p}) = \mathbb{E}[X_{1,e_j}(\Omega_{\mathbf{p}})\mathbf{1}_{\{U_j \in (p_j, p_j+\varepsilon]\}}] - \mathbb{E}[X_{0,e_j}(\Omega_{\mathbf{p}})\mathbf{1}_{\{U_j \in (p_j, p_j+\varepsilon]\}}] = \varepsilon \cdot \mathbb{E}_{\mathbf{p}}[\partial_{e_j} X]$$

We conclude

$$\frac{\partial}{\partial x_j} f(\mathbf{p}) = \lim_{\varepsilon \to 0} \frac{f(\mathbf{p} + \delta_j\varepsilon) - f(\mathbf{p})}{\varepsilon} = \mathbb{E}_{\mathbf{p}}[\partial_{e_j} X].$$

□

Chapter 8

The Sub-Critical Regime

8.1 Mean Component Size

An interesting quantity to consider is the typical size of a connected component in the percolation cluster. Recall that $\mathcal{C}(x)$ is the connected component of x. We will use the notation $\partial_n(x) = B(x,n) \setminus B(x,n-1)$ to denote the *sphere of radius* n; that is, all points at graph distance exactly n from x.

Definition 8.1.1 For $p \in (0,1)$ and p-percolation on a graph G, define

$$\chi_{G,x}(p) = \mathbb{E}_p[|\mathcal{C}(x)|].$$

✓ $\chi_{G,x}(p)$ may be infinite. It is immediate that $\chi_{G,x}(p) = \infty$ for all $p > p_c(G)$.

Suppose that

$$\mathbb{P}_p[x \leftrightarrow \partial_n(x)] \leq e^{-\lambda n}$$

for some $\lambda > 0$. Then,

$$\chi_x(p) = \sum_{n=0}^{\infty} \mathbb{E}_p[|\mathcal{C}(x) \cap \partial_n(x)|] \leq \sum_{n=0}^{\infty} |\partial_n(x)| \cdot e^{-\lambda n}.$$

So if the graph G is such that the spheres around x grow sub-exponentially (for example, this happens in $\mathbb{Z}^d$, where the balls grow polynomially), then $\chi_x(p) < \infty$.

The following was first proved by Hammersley. We prove it using a much simpler argument, that utilizes the BK inequality.

Lemma 8.1.2 Suppose G is a transitive graph (*e.g.* $\mathbb{Z}^d$). If $\chi(p) < \infty$ then the diameter of $\mathcal{C}(x)$ has an exponentially decaying tail. That is, there exists $\lambda > 0$ such that

$$\mathbb{P}_p[x \leftrightarrow \partial_n(x)] \leq e^{-\lambda n}.$$

Proof. The main observation is that for $n \geq m \geq 1$,

$$\{x \leftrightarrow \partial_{n+m}(x)\} \subset \bigcup_{y \in \partial_m(x)} \{x \leftrightarrow y\} \circ \{y \leftrightarrow \partial_n(y)\}.$$

Indeed, let $\omega \in \{x \leftrightarrow \partial_{n+m}(x)\}$. So there exists an open (in ω) path γ that goes from x to $\partial_{n+m}(x)$. By loop-erasing, we can get a simple path γ' that is a sub-path of γ (so is open in ω), and that goes from x to $\partial_{n+m}(x)$. If we take $y \in \partial_m(x)$ to be the first point of γ' on $\partial_m(x)$, then γ' is composed of two disjoint paths, one from x to y and one from y to $\partial_{n+m}(x)$. Since $\mathrm{dist}(x,y) = m$, there are two disjoint open (in ω) paths, one connecting x to y and one connecting y to distance at least n from y. So $\omega \in \{x \leftrightarrow y\} \circ \{y \leftrightarrow \partial_n(y)\}$.

We now apply the BK inequality to get for any $n \geq m \geq 1$,

$$\begin{aligned}\mathbb{P}_p[x \leftrightarrow \partial_{n+m}(x)] &\leq \sum_{y \in \partial_m(x)} \mathbb{P}_p[x \leftrightarrow y] \cdot \mathbb{P}_p[y \leftrightarrow \partial_n(y)] \\ &\leq \mathbb{E}_p[|\mathcal{C}(x) \cap \partial_m(x)|] \cdot \sup_{y \in \partial_m(x)} \mathbb{P}_p[y \leftrightarrow \partial_n(y)].\end{aligned}$$

Since

$$\sum_{m=0}^{\infty} \mathbb{E}_p[|\mathcal{C}(x) \cap \partial_m(x)|] = \chi(p) < \infty,$$

we have that there exists m large enough so that $\mathbb{E}_p[|\mathcal{C}(x) \cap \partial_m(x)|] \leq \frac{1}{2}$. Thus,

$$\sup_x \mathbb{P}_p[x \leftrightarrow \partial_{n+m}(x)] \leq \tfrac{1}{2} \cdot \sup_x \mathbb{P}_p[x \leftrightarrow \partial_n(x)].$$

We conclude that

$$\sup_x \mathbb{P}_p[x \leftrightarrow \partial_n(x)] \leq 2^{-\lfloor n/m \rfloor}.$$

□

If one examines the proof, it can be noted that we only required that for some $m \geq 1$, there exists a uniform (in x) bound on $\mathbb{E}_p[|\mathcal{C}(x) \cap \partial_m(x)|]$ which is strictly less than 1. Thus, we have actually shown,

Lemma 8.1.3 Let G be a graph. Consider percolation on G, and assume that there exists $\alpha < 1$ and $m \geq 1$ such that for all $x \in G$,

$$\mathbb{E}_p[|\mathcal{C}(x) \cap \partial_m(x)|] \leq \alpha.$$

Then, for any $x \in G$ and any n,

$$\mathbb{P}_p[x \leftrightarrow \partial_n(x)] \leq \alpha^{\lfloor n/m \rfloor}.$$

8.2 Menshikov and Aizenman-Barsky

Our goal is to prove that in the sub-critical regime, the probability of $x \leftrightarrow \partial_n(x)$ decays exponentially. As we have seen above, it suffices to show that for all $p < p_c$, the expected size of $\mathcal{C}(x)$ is finite.

A remark concerning critical points: Instead of p_c we could have defined the critical point

$$p_T = p_T(G) = \inf \{ p \ : \ \mathbb{E}_p[|\mathcal{C}(x)|] = \infty \} .$$

It is simple that $p_T \leq p_c$.

Menshikov and Aizenman & Barsky proved that for all $p < p_c$, $\mathbb{E}_p[|\mathcal{C}(x)|] < \infty$, which implies $p_T \geq p_c$, and consequently $p_T = p_c$. By Hammersley's argument above, this implies that there is exponential decay of the radius of $\mathcal{C}(x)$ in the sub-critical regime.

Menshikov's method is slightly less general than Aizenman-Barsky. With a subtle bootstrapping argument, Menshikov shows that for $p < p_c$, $\mathbb{P}_p[x \leftrightarrow \partial_n(x)] \leq \exp\left(-\lambda n (\log n)^{-2}\right)$. This implies that if $|\partial_n(x)| = O(e^{o(n(\log n)^{-2})})$ (for example, this holds in $\mathbb{Z}^d$) then $p_T = p_c$ and there is exponential decay of the radius tail.

Aizenman & Barsky's method is more general, and does not require growth assumptions on the graph metric. Moreover, it holds in a wider generality than percolation, although we do not detail that here.

Recently, Duminil-Copin & Tassion provided a more elementary and "cleaner" argument, which is in the next section.

We turn to the proof of the Aizenman-Barsky theorem.

8.3 Differential Inequalities

In order to understand the relation between p_c and the mean component size, we introduce the following augmentation of our percolation process:

Color the vertices of G randomly and independently, setting each vertex to be green with probability $\varepsilon > 0$, independent of the percolation. Let $\mathbb{P}_{p,\varepsilon}, \mathbb{E}_{p,\varepsilon}$ denote the probability measure and expectation with respect to the product of p-percolation and green vertices in G. We denote the random set of green vertices Γ.

We fix some root vertex $0 \in G$ and let $\mathcal{C} = \mathcal{C}(0)$. Also, let $\mathcal{C}_r(x) = \mathcal{C}(x) \cap B(0,r)$ and $\mathcal{C}_r = \mathcal{C}_r(0)$.

Define

$$\theta_r(p,\varepsilon) = \mathbb{P}_{p,\varepsilon}[\mathcal{C}_r \cap \Gamma \neq \emptyset],$$
$$\theta(p,\varepsilon) = \mathbb{P}_{p,\varepsilon}[\mathcal{C} \cap \Gamma \neq \emptyset] = \mathbb{P}_{p,\varepsilon}[0 \leftrightarrow \Gamma],$$
$$\chi(p,\varepsilon) = \mathbb{E}_{p,\varepsilon}[|\mathcal{C}| \cdot \mathbf{1}_{\{0 \not\leftrightarrow \Gamma\}}].$$

Exercise 8.1 Show that

$$\theta_r(p,\varepsilon) = 1 - \sum_{n=1}^{\infty} \mathbb{P}[|\mathcal{C}_r| = n](1-\varepsilon)^n,$$

$$\theta(p,\varepsilon) = 1 - \sum_{n=1}^{\infty} \mathbb{P}[|\mathcal{C}| = n](1-\varepsilon)^n,$$

$$\chi(p,\varepsilon) = \sum_{n=1}^{\infty} \mathbb{P}[|\mathcal{C}| = n] \cdot n(1-\varepsilon)^n.$$

Deduce that

$$\chi = (1-\varepsilon)\frac{\partial \theta}{\partial \varepsilon},$$

$$\frac{\partial \theta_r}{\partial \varepsilon} = \sum_{n=1}^{\infty} \mathbb{P}[|\mathcal{C}_r| = n] n (1-\varepsilon)^{n-1}.$$

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Exercise 8.2 Show that as $r \to \infty$,

$$\theta_r \nearrow \theta \qquad \text{and} \qquad \frac{\partial \theta_r}{\partial p} \to \frac{\partial \theta}{\partial p} \qquad \text{and} \qquad \frac{\partial \theta_r}{\partial \varepsilon} \to \frac{\partial \theta}{\partial \varepsilon}.$$

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Exercise 8.3 Prove that

$$\mathbb{P}[|\mathcal{C}_r \cap \Gamma| = 1] = \varepsilon \frac{\partial \theta_r}{\partial \varepsilon}.$$

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The most complicated estimate we have is the following:

Lemma 8.3.1 We have

$$\mathbb{P}[|\mathcal{C}_r \cap \Gamma| \geq 2] \leq (\theta_r)^2 + p \cdot \theta_{2r} \cdot \frac{\partial \theta_r}{\partial p}$$

Proof. For an edge e, we write $\mathcal{C}_{r,e}$ for the component of 0 in $B(0,r)$ when e is removed; that is $\mathcal{C}_{r,e}(\omega) = \mathcal{C}_r(\omega_{0,e})$. Let A_x be the event that $\mathcal{C}_r(x) \cap \Gamma \neq \emptyset$. So $\theta_r = \mathbb{P}[A_0]$. Also,

$$\mathbb{P}[A_x] \leq \mathbb{P}[\mathcal{C}(x) \cap B(x,2r) \cap \Gamma \neq \emptyset] = \theta_{2r}.$$

Step 1. Let $e = x \sim y$ be some edge in $B(x,r)$. The main effort is to bound from above the probability of the event $B_{xy} \cap A_y \circ A_y$ where $B_{xy} := \{x \in \mathcal{C}, \mathcal{C}_{r,e} \cap \Gamma = \emptyset\}$. On this event, we have that there must exist three *disjoint* open paths, one $\alpha : 0 \to x$, and the other two, $\beta, \gamma : y \to \Gamma$, such that β, γ do not intersect $\mathcal{C}_{r,e}$.

(*) Note that $B_{xy} \cap A_y \circ A_y$ implies that $\{e \text{ is pivotal for } \mathcal{C}_r \cap \Gamma \neq \emptyset\} \circ A_y$. We could bound this using Reimer's inequality, but let us circumvent this in order to be self contained.

Let Σ_e be the set of all subsets C of $B(x,r)$ such that $\mathbb{P}[B_{xy}, \mathcal{C}_{r,e} = C, A_y] >$

0. For any such $C \in \Sigma_e$, we have that

$$B_{xy} \cap \{\mathcal{C}_{r,e} = C\} \cap A_y \circ A_y = B_{xy} \cap \{\mathcal{C}_{r,e} = C\} \cap A_{C,y}(2)$$

where

$$A_{C,y}(2) = \Big\{ \text{ there exist } \textit{disjoint} \text{ open paths } \beta, \gamma : \Gamma \to y \text{ such that} \\ \text{both } \gamma, \beta \text{ do not intersect } E(C) \Big\}$$

Similarly,

$$B_{xy} \cap \{\mathcal{C}_{r,e} = C\} \cap A_y = B_{xy} \cap \{\mathcal{C}_{r,e} = C\} \cap A_{C,y}(1),$$

where

$$A_{C,y}(1) = \Big\{ \text{ there exists an open path } \beta : \Gamma \to y \text{ such that } \beta \text{ does not intersect } E(C) \Big\}$$

Thus, if we use $\mathbb{P}^C$ to denote the percolation measure on the subgraph $B(0,r) \setminus E(C)$ then, because the events $A_{C,y}(j)$ depend only on edges not in $E(C)$ and $B_{xy}, \{\mathcal{C}_{r,e} = C\}$ depend only on edges in $E(C)$, we have

$$\begin{aligned} \mathbb{P}[B_{xy}, \mathcal{C}_{r,e} = C, A_y \circ A_y] &= \mathbb{P}[B_{xy}, \mathcal{C}_{r,e} = C] \cdot \mathbb{P}[A_{C,y}(2)] \\ &= \mathbb{P}[B_{xy}, \mathcal{C}_{r,e} = C] \cdot \mathbb{P}^C[A_y \circ A_y] \\ &\leq \mathbb{P}[B_{xy}, \mathcal{C}_{r,e} = C] \cdot \mathbb{P}[A_{C,y}(1)] \cdot \mathbb{P}[A_y] \\ &\leq \mathbb{P}[B_{xy}, \mathcal{C}_{r,e} = C, A_y] \cdot \theta_{2r}, \end{aligned} \tag{8.1}$$

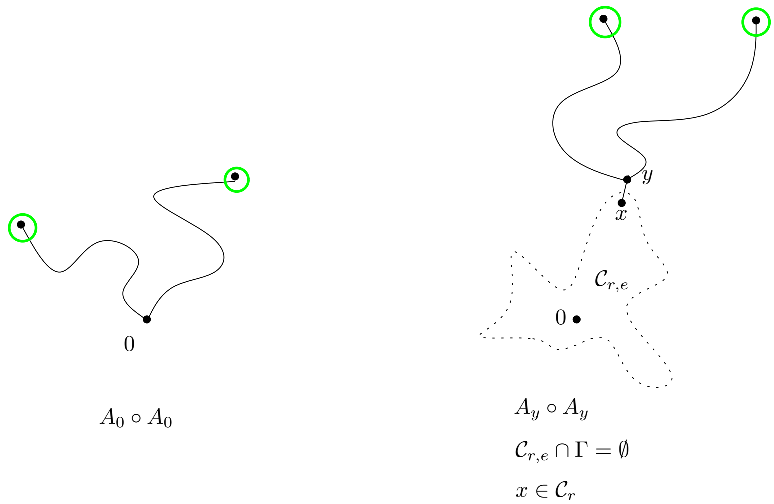
where we have used the BK inequality for the measure $\mathbb{P}^C$. Summing over all $C \in \Sigma_e$, we get that

$$\mathbb{P}[B_{xy}, A_y \circ A_y] \leq \theta_{2r} \cdot \mathbb{P}[B_{xy}, A_y].$$

The event $B_{xy} \cap A_y$ implies that e is pivotal for $\mathcal{C}_r \cap \Gamma \neq \emptyset$. Summing over possible choices for e, using Russo's formula we have

$$\mathbb{P}[\exists\ x \sim y \ : \ B_{xy}, A_y \circ A_y] \leq \theta_{2r} \cdot \frac{\partial \theta_r}{\partial p}. \tag{8.2}$$

Step 2. Now, consider the event that $|\mathcal{C}_r \cap \Gamma| \geq 2$ and $(A_0 \circ A_0)^c$. That is, 0 is connected to at least two vertices in Γ in $B(x,r)$, but there do not exist two disjoint paths from Γ to 0. Suppose that a, b are different two vertices



in $\Gamma \cap \mathcal{C}_r$. Let $\alpha : a \to 0$ be an open path. Note that if $b = 0$, then we would have the event $A_0 \circ A_0$ along the empty path and the path α. So we have assumed that $a, b \neq 0$.

Now, let e be the first edge along $\alpha : a \to 0$ such that removing e would disconnect 0 from Γ; *i.e.* $\mathcal{C}_{r,e} \cap \Gamma = \emptyset$. If such an edge does not exist, then removing the whole path $\alpha : a \to 0$, we still remain with 0 connected to Γ, which gives another disjoint path connecting Γ to 0. This contradicts our assumption that $(A_0 \circ A_0)^c$.

So we fix the first edge $e = x \sim y$ on $\alpha : a \to 0$ such that $\mathcal{C}_{r,e} \cap \Gamma = \emptyset$.

- By assumption, α is open, so e is open.
- e disconnects 0 from Γ so $\mathcal{C}_{r,e} \cap \Gamma = \emptyset$.
- $x \in \mathcal{C}_r$ and y is connected to Γ by the initial part of α
- Any path from Γ to 0 must pass through e first, so $A_y \circ A_y$.

We conclude that

$$\{|\mathcal{C}_r \cap \Gamma| \geq 2\} \cap (A_0 \circ A_0)^c \subset \{\exists\ e = x \sim y \ : \ e \text{ is open }, B_{xy}, A_y \circ A_y\}\,.$$

Note that the event $B_{xy} \cap A_y \circ A_y$ is independent of the state of e. So by

(8.2),

$$\mathbb{P}[|\mathcal{C}_r \cap \Gamma| \geq 2, (A_0 \circ A_0)^c] \leq p \cdot \theta_{2r} \cdot \frac{\partial \theta_r}{\partial p}.$$

The BK inequality gives that

$$\mathbb{P}[A_0 \circ A_0] \leq (\mathbb{P}[A_0])^2 \leq (\theta_r)^2,$$

which completes the lemma since

$$\mathbb{P}[|\mathcal{C} \cap \Gamma| \geq 2] \leq \mathbb{P}[A_0 \circ A_0] + \mathbb{P}[|\mathcal{C} \cap \Gamma| \geq 2, (A_0 \circ A_0)^c].$$

□

Lemma 8.3.2 If $\deg_G$ is the degree in G, then

$$(1-p)\frac{\partial \theta_r}{\partial p} \leq \deg_G (1-\varepsilon)\theta_{2r} \frac{\partial \theta_r}{\partial \varepsilon}.$$

Proof. As in Lemma 8.3.1, for an edge $e = x \sim y$ we consider the event $B_{xy} = \{x \in \mathcal{C}, \mathcal{C}_{r,e} \cap \Gamma = \emptyset\}$. Note that if e is pivotal for $\mathcal{C}_r \cap \Gamma \neq \emptyset$, then we must have that B_{xy}, A_y. By partitioning over possibilities for $B_{xy}, \mathcal{C}_{r,e} = C$, just as in (8.1) we have that

$$\begin{aligned}(1-p)\,\mathbb{P}[e \text{ is pivotal for } \mathcal{C}_r \cap \Gamma \neq \emptyset] &= \mathbb{P}[e \text{ is closed and pivotal for } \mathcal{C}_r \cap \Gamma \neq \emptyset] \\ &\leq \mathbb{P}[e \text{ is closed }, B_{xy}, A_y] \leq \mathbb{P}[e \text{ is closed }, B_{xy}] \cdot \theta_{2r}.\end{aligned}$$

If e is closed, then $\mathcal{C}_{r,e} = \mathcal{C}_r$, so summing over e with Russo's formula,

$$\begin{aligned}(1-p)\frac{\partial \theta_r}{\partial p} &\leq \theta_{2r} \cdot \sum_{e=x\sim y} \mathbb{P}[e \text{ is closed }, B_{xy}] \\ &\leq \theta_{2r} \cdot \sum_{x\sim y} \mathbb{P}[x \in \mathcal{C}_r, \mathcal{C}_r \cap \Gamma = \emptyset] \leq \theta_{2r} \cdot \deg_G \cdot \mathbb{E}[|\mathcal{C}_r| \cdot \mathbf{1}_{\{\mathcal{C}_r \cap \Gamma = \emptyset\}}].\end{aligned}$$

The proof is completed since

$$\mathbb{E}[|\mathcal{C}_r| \cdot \mathbf{1}_{\{\mathcal{C}_r \cap \Gamma = \emptyset\}}] = \sum_{n=1}^{\infty} \mathbb{P}[|\mathcal{C}_r| = n] n (1-\varepsilon)^n = (1-\varepsilon)\frac{\partial \theta_r}{\partial \varepsilon}.$$

□

To summarize this section, combining Exercise 8.3 and Lemmas 8.3.1 and 8.3.2, we have the following differential inequalities.

Lemma 8.3.3 For $\deg_G$ the degree in G,

$$\theta \leq \varepsilon \frac{\partial \theta}{\partial \varepsilon} + \theta^2 + p \cdot \theta \frac{\partial \theta}{\partial p}$$

and

$$(1-p)\frac{\partial \theta}{\partial p} \leq \deg_G (1-\varepsilon) \cdot \theta \frac{\partial \theta}{\partial \varepsilon}$$

Proof. Use

$$\mathbb{P}[\mathcal{C}_r \cap \Gamma \neq \emptyset] \leq \mathbb{P}[|\mathcal{C}_r \cap \Gamma| = 1] + \mathbb{P}[|\mathcal{C}_r \cap \Gamma| \geq 2].$$

Then, combine Exercise 8.3 and Lemmas 8.3.1 and 8.3.2, and send $r \to \infty$. □

8.4 The Aizenman-Barsky Theorem

Theorem 8.4.1 Let $p < p_c(G)$ for a transitive graph G. Then,

$$\mathbb{E}_p[|\mathcal{C}(x)|] < \infty.$$

Proof. Let $p < q < p_c$. Note that for any $s \in [p, q]$, $\theta(s, \varepsilon) \leq \theta(q, \varepsilon) \to 0$ as $\varepsilon \to 0$. Let $\eta > 0$ be small enough so that for all $s \in [p, q]$ and $\varepsilon < \eta$, $\theta(s, \varepsilon) \leq \frac{1}{2}$.

Recall that

$$\tfrac{\partial \theta}{\partial \varepsilon}(s, \varepsilon) = \chi(s, \varepsilon)(1-\varepsilon)^{-1} \geq \mathbb{P}[|\mathcal{C}| = 1] \geq (1-p)^{\deg_G} > 0.$$

So θ is strictly increasing in ε, and so we can invert it on $[0, \eta]$ and write $f_s(x) = \varepsilon$ for the unique ε such that $\theta(s, \varepsilon) = x$. This implies that $f_s'(\theta(s, \varepsilon)) \cdot \frac{\partial \theta}{\partial \varepsilon}(s, \varepsilon) = 1$, for all $\varepsilon < \eta$ and $s \in [p, q]$. Thus, for all $s \in [p, q]$ and $x \leq \frac{1}{2}$, we get that $f_s'(x) \leq (1-q)^{-\deg_G} < \infty$.

Recall our differential inequalities from Lemma 8.3.3. Combining them both we have that for $C = \sup_{s \in [p,q]} \frac{s \deg_G}{1-s} = \frac{q \deg_G}{1-q}$,

$$\theta \leq \frac{\partial \theta}{\partial \varepsilon} \cdot \left(\varepsilon + C\theta^2\right) + \theta^2,$$

For $x \leq \frac{1}{2}$ this is the same as

$$x \leq (f_s'(x))^{-1} \cdot \left(f_s(x) + Cx^2\right) + x^2$$

which becomes

$$\tfrac{1}{x} f_s'(x) - \frac{f_s(x)}{x^2} \leq C + f_s'(x).$$

Since f_s' is uniformly bounded by $(1-q)^{-\deg_G}$ we obtain

$$\frac{d}{dx}\left(\tfrac{1}{x} f_s\right)(x) = \tfrac{1}{x} f_s'(x) - \frac{f_s(x)}{x^2} \leq C(q).$$

Integrating over an interval $[y, x]$ for small enough $y < x$ (in fact $x \leq \frac{1}{2}$), we have that $\frac{1}{x} f_s(x) - \frac{1}{y} f_s(y) \leq C(q)(x - y)$.

Now, assume for a contradiction that $\mathbb{E}_p[|\mathcal{C}(x)|] = \infty$. Since $\theta(p) = 0$ this implies that

$$f_p'(0) = \lim_{\varepsilon \to 0} f_p'(\theta(p, \varepsilon)) = \lim_{\varepsilon \to 0} \frac{1 - \varepsilon}{\chi(p, \varepsilon)} = 0.$$

Thus, we may send $y \to 0$ and get $\frac{1}{x} f_p(x) \leq C(q)x$, which is to say that for all $\varepsilon < \eta$,

$$\theta(p, \varepsilon) \geq \sqrt{\tfrac{\varepsilon}{C(q)}}.$$

Thus, for any $\varepsilon < \eta$,

$$\limsup_{\varepsilon' \to 0} \frac{\log \frac{\theta(q,\varepsilon)}{\theta(p,\varepsilon')}}{\log \frac{\varepsilon}{\varepsilon'}} \leq \frac{1}{2} \cdot \limsup_{\varepsilon' \to 0} \frac{\log \frac{C(q)}{\varepsilon'}}{\log \frac{\varepsilon}{\varepsilon'}} = \frac{1}{2}.$$

We use this in the first inequality

$$\theta \leq \varepsilon \frac{\partial \theta}{\partial \varepsilon} + \theta^2 + p\theta \frac{\partial \theta}{\partial p},$$

which may be re-written as

$$0 \leq \frac{\partial}{\partial \varepsilon} \log \theta + \frac{1}{\varepsilon} \cdot \frac{\partial}{\partial p} \left(p\theta - p\right).$$

We now integrate this inequality over $[p, q]$ and $[\varepsilon', \varepsilon]$ for $0 < \varepsilon' < \varepsilon < \eta$. We use the fact that $\theta(s, \xi)$ is maximized on $[p, q] \times [\varepsilon', \varepsilon]$ at $(s, \xi) = (q, \varepsilon)$

and minimized at $(s,\xi)=(p,\varepsilon')$.

$$0 \le \int_p^q \log \frac{\theta(s,\varepsilon)}{\theta(s,\varepsilon')} ds + \int_{\varepsilon'}^{\varepsilon} \frac{q\theta(q,\xi)-q-p\theta(p,\xi)+p}{\xi} d\xi$$
$$\le (q-p)\log\frac{\theta(q,\varepsilon)}{\theta(p,\varepsilon')} + \big(q\theta(q,\varepsilon)-p\theta(p,\varepsilon')+p-q\big)\cdot\log\tfrac{\varepsilon}{\varepsilon'}.$$

Dividing by $\log\frac{\varepsilon}{\varepsilon'}$, we get taking $\varepsilon'\to 0$ that

$$0 \le \tfrac{1}{2}(q-p) + q\theta(q,\varepsilon) - p\theta(p,0) + p - q = -\tfrac{1}{2}(q-p) + q\theta(q,\varepsilon).$$

Another limit as $\varepsilon \to 0$ gives that

$$0 < \tfrac{1}{2q}(q-p) \le \theta(q,0) = \theta(q),$$

a contradiction! □

8.5 The Duminil-Copin & Tassion argument

Quite recently, Duminil-Copin & Tassion provided a much more elementary argument for the exponential decay of long connections (what is sometimes called *sharpness of the phase transition*).

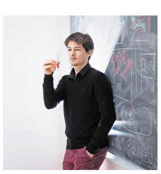
Recall the definition of *connections inside a given set*: For a subset $S \subset V$ we write $x \overset{S}{\leftrightarrow} y$ (resp. $A \overset{S}{\leftrightarrow} B$) if there exists an open path, only using edges contained in S, that connects x,y (resp. some $x\in A, y \in B$).

Hugo Duminil-Copin

Also, recall the *edge boundary* of S, defined as

$$\partial_e S = \{(x,y) \ : \ S \ni x \sim y \notin S\}.$$

Let $S \subset V$ be a finite subset containing o. Define



Vincent Tassion

$$\varphi_p(S) = p \sum_{(x,y)\in\partial_e S} \mathbb{P}_p[o \overset{S}{\leftrightarrow} x].$$

So $\varphi_p(S)$ is p times the expected number of edges in the edge boundary $\partial_e S$ of S, that are connected to o inside S.

Let $B_r = B_r(o)$ be the ball of radius $r>0$ around o in the graph. Let $r>0$ be large enough so that $S \cup \partial S \subset B_r$. Thus, if $y \sim S$ then $y \in B_r$.

For all $r > 0$ define

$$\chi(r,p) = \sum_{x \in B_r} \mathbb{P}_p[o \overset{B_r}{\leftrightarrow} x].$$

Exercise 8.4 Show that for any $p \in [0,1]$,

$$\lim_{r \to \infty} \chi(r,p) = \chi(p) = \mathbb{E}_p \, |\mathcal{C}(o)|.$$

◇ ◇ ◇

Lemma 8.5.1 Let A, B, S be subsets such that $o \in S$, $B \cap S = \emptyset$ and $|S| < \infty$. Then,

$$\mathbb{P}_p[o \overset{A}{\leftrightarrow} B] \leq \sum_{(x,y) \in \partial_e S} p \cdot \mathbb{P}_p[o \overset{S}{\leftrightarrow} x] \cdot \mathbb{P}_p[y \overset{A}{\leftrightarrow} B].$$

Proof. If $o = x_0, x_1, \ldots, x_t \in B$ is an open path contained in A from o to B (*i.e.* all edges $x_i \sim x_{i+1}$ are open and all $x_i \in A$), then for k being the first index for which $x_{k+1} \not\in S$ we have that $o \overset{S}{\leftrightarrow} x_k$, and $x_k \sim x_{k+1}$ is open, and $x_{k+1} \overset{A}{\leftrightarrow} B$. Moreover, all these events are disjointly occurring as may be witnessed by the disjoint sets of the edges of $\{x_0, \ldots, x_k\}, \{x_k, x_{k+1}\}, \{x_{k+1}, \ldots, x_t\}$. In other words,

$$\{o \overset{A}{\leftrightarrow} B\} \subset \bigcup_{(x,y) \in \partial_e S} \{o \overset{S}{\leftrightarrow} x\} \circ \{y \overset{A}{\leftrightarrow} B\} \circ \{x \sim y \text{ is open }\}.$$

The conclusion follows by the BK inequality, □

Lemma 8.5.2 Let G be a transitive graph. Suppose that $o \in S$ for some finite subset S. If $\varphi_p(S) < 1$ then $\mathbb{E}_p \, |\mathcal{C}(o)| \leq \frac{|S|}{1-\varphi_p(S)} < \infty$.

Proof. Let S be a finite subset such that $o \in S, \varphi_p(S) < 1$.

For any vertex y define

$$\chi_{y;o}(r,p) = \sum_{v \in B_r(o)} \mathbb{P}_p[y \overset{B_r(o)}{\leftrightarrow} v].$$

Note that $\chi_{y;o}(r,p) = 0$ if $y \notin B_r(o)$. For any r let $u_r \in B_r(o)$ be the vertex for which $\chi_{u_r;o}(r,p) = \max_y \chi_{y;o}(r,p)$. Of course, $\chi_{u_r;o}(r,p) \geq \chi_{o;o}(r,p) = \chi(r,p) \to \chi(p)$.

For every vertex u let ψ_u be an automorphism of G mapping o to u (recall that G is assumed to be transitive). If $v \in B_r(o) \setminus \psi_u S$ then by Lemma 8.5.1,

$$\mathbb{P}_p[u \overset{B_r(o)}{\leftrightarrow} v] = \sum_{(x,y)\in\partial_e \psi_u S} p \cdot \mathbb{P}_p[u \overset{\psi_u S}{\leftrightarrow} x] \cdot \mathbb{P}_p[y \overset{B_r(o)}{\leftrightarrow} v],$$

so that summing over $v \in B_r(o) \setminus \psi_u S$ we have

$$\sum_{v\in B_r(o)\setminus\psi_u S} \mathbb{P}_p[u \overset{B_r(o)}{\leftrightarrow} v] \leq \varphi_p(S) \cdot \max_y \chi_{y;o}(r,p) \leq \varphi_p(S) \cdot \chi_{u_r;o}(r,p).$$

Since $\mathbb{P}_p[u \overset{B_r(o)}{\leftrightarrow} v] \leq 1$ always, we have altogether that

$$\chi_{u;o}(r,p) = \sum_{v\in B_r(o)} \mathbb{P}_p[u \overset{B_r(o)}{\leftrightarrow} v] \leq |S| + \varphi_p(S) \cdot \chi_{u_r;o}(r,p).$$

Choosing $u = u_r$ we arrive at the inequality

$$\chi(r,p) \leq \chi_{u_r;o}(r,p) \leq \frac{|S|}{1-\varphi_p(S)}.$$

Taking $r \to \infty$ proves the lemma. $\square$

Lemma 8.5.3 For any $r > 0$ and $p \in (0,1)$,

$$\tfrac{\partial}{\partial p} \mathbb{P}_p[o \leftrightarrow \partial B_r] \geq \frac{1}{p(1-p)} \cdot \inf_{o\in S\subset B_r} \varphi_p(S) \cdot \Big(1 - \mathbb{P}_p[o \leftrightarrow \partial B_r]\Big).$$

Proof. Consider the event $\{o \leftrightarrow \partial B_r\}$. Let P be the (random) set of pivotal edges for this event. Also, set $\mathcal{C} = \{x \in B_r \ : \ x \not\leftrightarrow \partial B_r\}$. One notes that if $\mathcal{C} = S$ then $P = \{\{x,y\} \in E \ : \ (x,y) \in \partial_e S \ : \ x \overset{S}{\leftrightarrow} o\}$. Also, $o \not\leftrightarrow \partial B_r$ if and only if $o \in \mathcal{C} \subset B_r$. Let $E = \{e \in E(G) \ : \ e \cap B_r \neq \emptyset\}$ be the set of edges intersecting the ball of radius r. So $P \subset E$ always. We compute:

$$\begin{aligned}\sum_{e\in E} \mathbb{P}_p[e \in P \ , \ o \not\leftrightarrow \partial B_r] &= \sum_{e\in E} \sum_{o\in S\subset B_r} \mathbb{P}_p[e \in P \ , \ \mathcal{C} = S] \\ &= \sum_{o\in S\subset B_r} \sum_{(x,y)\in\partial_e S} \mathbb{P}_p[o \overset{S}{\leftrightarrow} x \ , \ \mathcal{C} = S].\end{aligned}$$

When $S \subset B_r$, the event $\mathcal{C} = S$ depends only on edges not contained in S. So the events $\{o \overset{S}{\leftrightarrow} x\}, \{\mathcal{C} = S\}$ are independent. Thus,

$$\begin{aligned}\sum_{e \in E} \mathbb{P}_p[e \in P \ , \ o \not\leftrightarrow \partial B_r] &= \sum_{o \in S \subset B_r} \sum_{(x,y) \in \partial_e S} \mathbb{P}_p[o \overset{S}{\leftrightarrow} x] \cdot \mathbb{P}_p[\mathcal{C} = S] \\ &= \sum_{o \in S \subset B_r} \mathbb{P}_p[\mathcal{C} = S] \cdot \tfrac{1}{p} \varphi_p(S) \\ &\geq \mathbb{P}_p[o \not\leftrightarrow \partial B_r] \cdot \tfrac{1}{p} \inf_{o \in S \subset B_r} \varphi_p(S).\end{aligned}$$

The proof is now complete since Russo's formula tells us that

$$\sum_{e \subset B_r} \mathbb{P}_p[e \in P \ , \ o \not\leftrightarrow \partial B_r] = (1-p) \tfrac{\partial}{\partial p} \mathbb{P}_p[o \leftrightarrow \partial B_r],$$

where we have used that $o \leftrightarrow \partial B_r$ is increasing. □

Alternative proof of the Aizenman-Barsky Theorem. Define

$$q_c = \sup\{p \ : \ \exists \ S \subset V \ , \ |S| < \infty \ , \ \varphi_p(S) < 1\}.$$

Lemmas 8.1.2 and 8.5.2 show that if there exists a finite set $S \subset V$ containing o such that $\varphi_p(S) < 1$, then $\mathbb{P}_p[o \leftrightarrow \partial B_r]$ decays exponentially. This implies that $q_c \leq p_c$.

We now show that for any $p > q_c$ we have $\theta(p) > 0$, so $q_c = p_c$.

Let $f(p) = \mathbb{P}_p[o \not\leftrightarrow \partial B_r]$. By Lemma 8.5.3, if $p > q_c$ then

$$\tfrac{\partial}{\partial p} \log f(p) \leq -\tfrac{1}{p(1-p)}.$$

Thus, integrating over (q_c, p) we have

$$\log \tfrac{f(q_c)}{f(p)} \geq \int_{q_c}^{p} \tfrac{1}{\xi(1-\xi)} d\xi = \log \tfrac{p(1-q_c)}{q_c(1-p)},$$

which implies that $f(p) \leq \frac{q_c}{1-q_c} \cdot \frac{1-p}{p}$. Hence,

$$\mathbb{P}_p[o \leftrightarrow \partial B_r] \geq 1 - \tfrac{q_c}{1-q_c} \cdot \tfrac{1-p}{p} = \frac{p(1-q_c) - q_c(1-p)}{p(1-q_c)} = \frac{p - q_c}{p(1-q_c)}.$$

For any $p > q_c$, the left hand side is positive independent of r, so taking $r \to \infty$, $\theta(p) \geq \frac{p-q_c}{p(1-q_c)} > 0$ for any $p > q_c$. □

The behavior of χ at p_c can also be deduced. The next result was first noted by Aizenman & Newman.

Proposition 8.5.4 At $p = p_c$ we have that $\chi(p_c) = \infty$. (Thus, $p < p_c$ if and only if $\chi(p) < \infty$.)

Proof. In the Duminil-Copin & Tassion proof of the Aizenman-Barsky Theorem we showed that

$$p_c = q_c = \sup\{p \ : \ \exists\, o \in S \subset V \ , \ |S| < \infty \ , \ \varphi_p(S) < 1\}.$$

The set $\Phi = \{p \ : \ \exists\, o \in S \subset V \ , \ |S| < \infty \ , \ \varphi_p(S) < 1\}$ is an open set. Indeed, since

$$X = \sum_{(x,y) \in \partial_e S} \mathbf{1}_{\left\{o \overset{S}{\leftrightarrow} x\right\}}$$

is a random variable depending on only finitely many edges (only on the edges in S and $\partial_e S$), we have that $p \cdot \mathbb{E}_p[X] = \varphi_p(S)$ is a continuous function of p. So

$$\Phi = \bigcup_{\substack{|S|<\infty \\ o \in S}} \{p \ : \ \varphi_p(S) < 1\}$$

is open as a union of open sets.

This implies that $p_c = q_c = \sup \Phi \not\in \Phi$. Specifically, $\varphi_{p_c}(S) \geq 1$ for any finite $S \ni o$.

We now have

$$\infty \leq \sum_r \varphi_{p_c}(B_r(o)) = \sum_r \sum_{(x,y) \in \partial_e B_r(o)} p_c \cdot \mathbb{P}_{p_c}[o \overset{B_r(o)}{\leftrightarrow} x] \leq \mathbb{E}_{p_c} |\mathcal{C}(o)| \cdot \deg_G$$

$\square$

8.6 Summary

Just to summarize, in this chapter we gave a few characterizations for p_c in transitive graphs.

Theorem 8.6.1 Let G be a transitive graph. For a finite subset S containing a vertex o set $\varphi_p(S) = \sum_{(x,y)\in\partial_e S} p\,\mathbb{P}_p[o \overset{S}{\leftrightarrow} x]$. Then, $p_c = p_T = q_c$ where

$$\begin{aligned} p_c &= \sup\{p \ : \ \theta(p) = 0\} \\ p_T &= \sup\{p \ : \ \chi(p) < \infty\} \\ q_c &= \sup\{p \ : \ \exists\, o \in S \subset V \ , \ |S| < \infty \ , \ \varphi_p(S) < 1\} \end{aligned}$$

Moreover, at $p = p_c$ we have that $\chi(p_c) = \infty$ and $\varphi_{p_c}(S) \geq 1$ for any finite S. The question if $\theta(p_c) = 0$ is one of the most important open questions in probability.

We have also seen that $\chi(p) < \infty$ implies that $\mathbb{P}_p[o \leftrightarrow \partial B_r(o)]$ decays exponentially. The phenomena that sub-critical long connections decay exponentially is sometimes called *sharpness of the phase transition*.

Chapter 9

Planar Percolation

9.1 Duality

Let G be a graph drawn in the plane. The dual of G, denoted $\hat{G}$ is the graph whose vertices are the faces of G (including the infinite face if it exists). Two faces are adjacent if they share an edge, so to every edge e in G we have assigned a dual edge $\hat{e}$ in $\hat{G}$.

If $\omega \in \{0,1\}^{E(G)}$ then define the dual of ω to be

$$\hat{\omega}(\hat{e}) = 1 - \omega(e).$$

That is, a dual edge is open if and only if the corresponding primal edge is closed.

We have already used duality to prove an upper bound for $p_c(\mathbb{Z}^2)$.

The following property of planar graphs is the key to most of the arguments in planar percolation. It is not simple to rigorously write down the proof, although the proof is pretty clear from the figure.

Proposition 9.1.1 Let G be a planar graph. Let $x \sim y$ be a specific edge in $E(G)$. Let $\hat{x} \sim \hat{y}$ be the dual of $x \sim y$. Let $\omega \in \{0,1\}^{E(G)}$ be any percolation configuration, and $\hat{\omega}$ its dual. Let G' be the graph G with the edge $x \sim y$ removed and let $\hat{G}'$ be the graph $\hat{G}$ with the edge $\hat{x} \sim \hat{y}$ removed. Then, either there exists a path $\alpha : x \to y$ in G' that admits $\omega(\varepsilon) = 1$ for all edges $\varepsilon \in \alpha$, or there exists a path $\beta : \hat{x} \to \hat{y}$ in $\hat{G}'$ that admits $\hat{\omega}(\hat{\varepsilon}) = 1$ for all dual edges $\hat{\varepsilon} \in \beta$.

Let us prove a simpler version of this proposition, for $\mathbb{Z}^2$.

✓ NOTATION: A rectangle in $\mathbb{Z}^2$ is a set of the form $[a, a'] \times [b, b'] \cap \mathbb{Z}^2$. If R is a rectangle, then $\partial_i R$, the *inner boundary of* R, is the set of vertices just inside R; *i.e.* ,

$$\partial_i R = \{x \in R \ : \ \exists \ y \notin R \ , \ x \sim y\} \, .$$

We also find it convenient to define the N, S, E, W boundaries: If $R = [a, a'] \times [b, b'] \cap \mathbb{Z}^2$ where a, a', b, b' are all integers,

$$\partial_{\mathsf{N}} R = [a, a'] \times \{b'\} \cap \mathbb{Z}^2 \qquad \partial_{\mathsf{E}} R = \{a'\} \times [b, b'] \cap \mathbb{Z}^2,$$

and similarly for south and west.

For a rectangle R, we write $(\leftrightarrow R)$ for the event that there is an open path crossing R from left to right; that is, the event that there exist $x \in \partial_{\mathsf{W}} R, y \in \partial_{\mathsf{E}} R$, and a path $\gamma : x \to y$ inside R that is open. Similarly, $(\updownarrow R)$ denotes the event that there is a crossing from top to bottom of R, that is from $\partial_{\mathsf{N}} R$ to $\partial_{\mathsf{S}} R$.

There are also two possible duals for a rectangle R: For a rectangle $R = [a, a'] \times [b, b'] \cap \mathbb{Z}^2$ integers, let R^h be the horizontal dual, which is the subgraph of $\widehat{\mathbb{Z}^2} = (\frac{1}{2}, \frac{1}{2}) + \mathbb{Z}^2$, induced on the vertices

$$R^h = [a - \tfrac{1}{2}, b + \tfrac{1}{2}] \times [a' + \tfrac{1}{2}, b' - \tfrac{1}{2}] \cap \widehat{\mathbb{Z}^2}.$$

Similarly, the vertical dual is

$$R^v = [a + \tfrac{1}{2}, b - \tfrac{1}{2}] \times [a' - \tfrac{1}{2}, b' + \tfrac{1}{2}] \cap \widehat{\mathbb{Z}^2}.$$

The main duality observation is:

Proposition 9.1.2 Let R be a rectangle in $\mathbb{Z}^2$. Let ω be any percolation configuration. Then, exactly one of the following holds: either $\omega \in (\leftrightarrow R)$ or $\hat{\omega} \in (\updownarrow R^v)$.

Example 9.1.3 Let R be the rectangle $[0, n+1] \times [0, n] \cap \mathbb{Z}^2$.

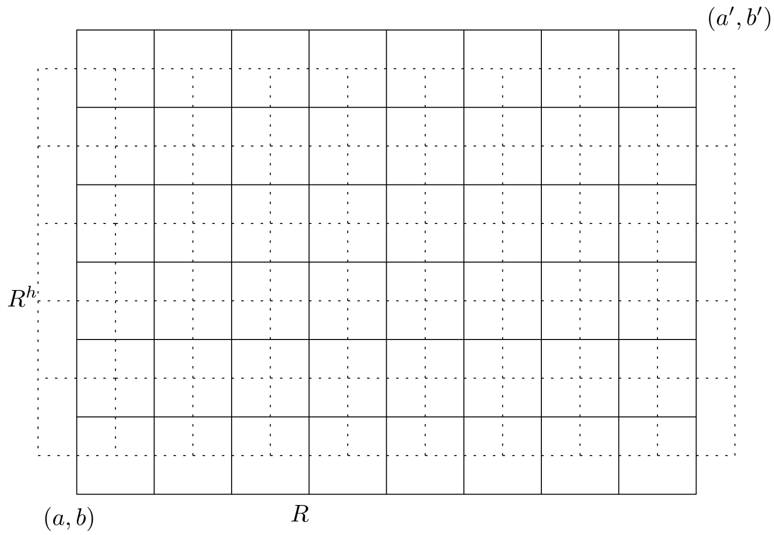


Figure 9.1: A rectangle in $\mathbb{Z}^2$ and its dual.

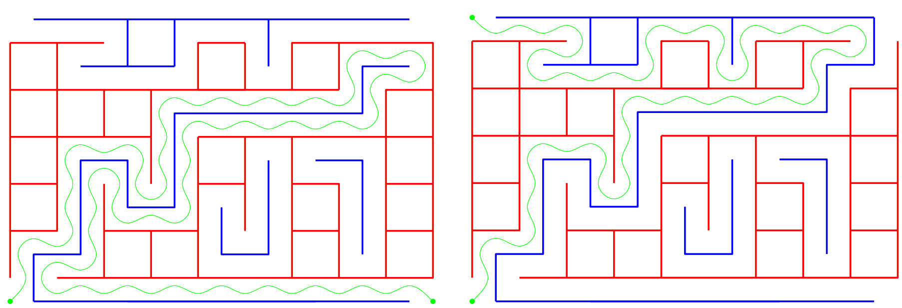


Figure 9.2: Two possibilities for exploration of the component of $\partial_{\mathsf{W}} R$.

Note that R^v is a rotated version of R, and the symmetry gives that

$$\mathbb{P}_p[\leftrightarrow R] = \mathbb{P}_{1-p}[\updownarrow R^v].$$

Since these events are a partition of the whole space,

$$\mathbb{P}_p[\leftrightarrow R] + \mathbb{P}_{1-p}[\leftrightarrow R] = \mathbb{P}_p[\leftrightarrow R] + \mathbb{P}_p[\updownarrow R^v] = 1.$$

Taking $p = 1/2$ gives $\mathbb{P}_{1/2}[\leftrightarrow R] = \frac{1}{2}$. △▽△

9.2 Zhang's Argument and Harris' Theorem

The following theorem was first proved by Harris in 1960. Zhang provided a simple (and more general) argument later on.

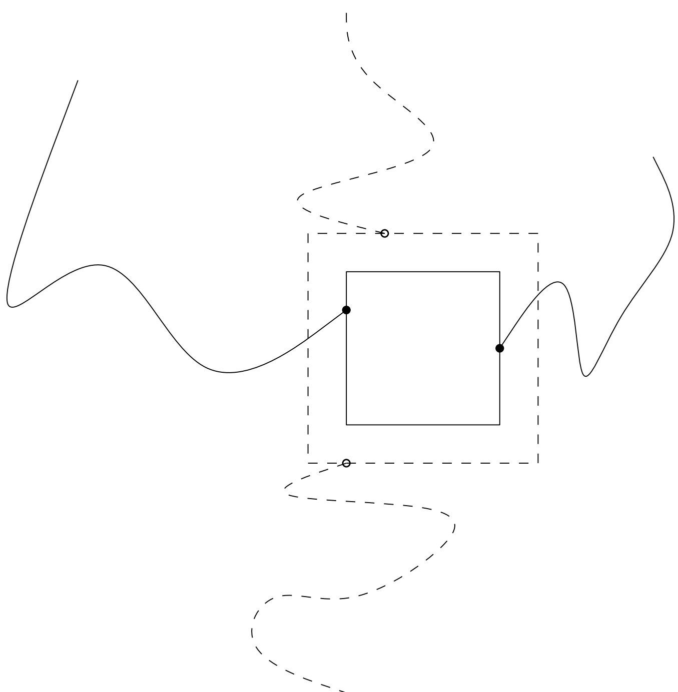


Figure 9.3: The event A from Zhang's argument.

Theorem 9.2.1 For bond percolation on $\mathbb{Z}^2$, $\theta(1/2) = 0$. Consequently, $p_c(\mathbb{Z}^2) \geq 1/2$.

Proof. Assume for a contradiction that $\theta(1/2) > 0$. So $\mathbb{P}_{1/2}$-a.s. there exists a unique infinite component.

Let R be the $n \times n$ rectangle $[-n, n] \times [-n, n] \cap \mathbb{Z}^2$. Take n large enough so that R intersects the infinite component with probability at least $1 - 5^{-4}$ (as $n \to \infty$ this probability tends to 1).

Now, consider the event

$$A_E = \left\{ \omega \ : \ \omega_{0,E(R)} \in \{ \partial_{\mathsf{E}} R \leftrightarrow \infty \} \right\}.$$

That is, A_E is the event that $\partial_{\mathsf{E}} R$ is connected to the infinite component by an open path that does not pass through R. Define A_N, A_S, A_W similarly. The rotation-by-$\pi/2$ symmetry implies that all these events have the same probability. Moreover, since these are all increasing events, by Harris' Lemma,

$$\begin{aligned} 1 - 5^{-4} &\leq \mathbb{P}_{1/2}[R \leftrightarrow \infty] \leq \mathbb{P}_{1/2}[A_N \cup A_S \cup A_E \cup A_W] \\ &= 1 - \mathbb{P}_{1/2}[A_N^c \cap A_S^c \cap A_E^c \cap A_W^c] \leq 1 - (1 - \mathbb{P}_{1/2}[A_E])^4 \end{aligned}$$

So $\mathbb{P}_{1/2}[A_E] = \mathbb{P}_{1/2}[A_W] \geq \frac{4}{5}$.

Now, let R' be the dual rectangle containing R; that is $R' = [-\frac{1}{2}, n + \frac{1}{2}] \times [-\frac{1}{2}, n + \frac{1}{2}] \cap \widehat{\mathbb{Z}^2}$. Define the events A'_N, A'_S, A'_E, A'_W analogously to the above, for the rectangle R' in the dual graph $\widehat{\mathbb{Z}^2}$. Again we have that $\mathbb{P}_{1/2}[A'_N] = \mathbb{P}_{1/2}[A'_S] \geq \frac{4}{5}$.

Finally, consider the event $A = A_E \cap A_W \cap A'_N \cap A'_S$. We have that

$$\mathbb{P}[A^c] \leq \mathbb{P}[A_E^c] + \mathbb{P}[A_W^c] + \mathbb{P}[(A'_N)^c] + \mathbb{P}[(A'_S)^c] \leq \tfrac{4}{5},$$

so $\mathbb{P}_{1/2}[A] \geq \frac{1}{5} > 0$.

Moreover, the event A does not depend on edges inside R. Also, if we open all edges in R, then together with the event A this implies that there exist at least two infinite components in $\widehat{\mathbb{Z}^2}$. This is because the union of $E(R)$ and the infinite paths from A_E, A_W is an open *connected* set of primal edges that separates the whole plane into two components. Since the dual infinite paths from A'_N, A'_S start in different components of this separation, they cannot connect in the dual graph. Otherwise, somewhere a dual open edge would cross a primal open edge which is impossible by construction.

That is,

$$\begin{aligned} 0 < 2^{-|E(R)|} \cdot \tfrac{1}{5} &= \mathbb{P}_{1/2}[A, E(R) \text{ is open }] \\ &\leq \mathbb{P}_{1/2}[\text{ there are two infinite components in the dual } \widehat{\mathbb{Z}^2}] \end{aligned}$$

But this last event must have probability 0 by the Burton-Keane Theorem.

□

9.3 Kesten's Theorem - A Modern Proof

We are now in position to prove Kesten's remarkable theorem from 1980:

Theorem 9.3.1 Consider bond percolation on $\mathbb{Z}^2$. Then, for any $p > \frac{1}{2}$, $\theta(p) > 0$. Consequently, $p_c(\mathbb{Z}^2) = \frac{1}{2}$.

Proof. Let R be the $n \times (n+1)$ rectangle, $R = [0, n+1] \times [0, n] \cap \mathbb{Z}^2$. We know by symmetry that $\mathbb{P}_{1/2}[\leftrightarrow R] = \frac{1}{2}$, and specifically, uniformly bounded away from 0 (as $n \to \infty$).

Suppose that $\frac{1}{2} < p_c$. Then, by Aizenman-Barsky (5kg hammer to kill a fly!), $\mathbb{E}_{1/2}[|\mathcal{C}(0)|] < \infty$, and

$$\mathbb{P}_{1/2}[x \leftrightarrow \partial_n(x)] \leq e^{-cn},$$

uniformly in x, for all n. Thus,

$$\mathbb{P}_{1/2}[\leftrightarrow R] \leq \sum_{x \in \partial_{\mathsf{W}} R} \mathbb{P}_{1/2}[x \leftrightarrow \partial_{\mathsf{E}} R] \leq \sum_{x \in \partial_{\mathsf{E}} R} \mathbb{P}_{1/2}[x \leftrightarrow \partial_n(x)] \leq n e^{-cn} \to 0.$$

This contradicts $\mathbb{P}_{1/2}[\leftrightarrow R] = \frac{1}{2}$. □

Kesten's original proof that $p_c(\mathbb{Z}^2) \leq 1/2$ did not use Aizenman-Barsky, which came later.

9.4 Site Percolation and the Triangular Lattice

For pedagogical reasons, we move now to site percolation on the triangular lattice.

In site percolation, the sample space is configuration of open and closed vertices: $\{0,1\}^G$. The notations for connections, components, etc. are the same, and we will add a super-script $\mathbb{P}^{\mathsf{s}}_p, \mathbb{E}^{\mathsf{s}}_p$ to specify site-percolation if there is possibility of confusion.

We leave it to the reader to verify that all the general results apply to site percolation as well: namely, Harris' Lemma, BK inequality, Russo's formula, and the Aizenman-Barsky Theorem.

As for the triangular lattice, it is the lattice whose faces are equilateral triangles. For definiteness, let us fix $\mathbb{T}$ to be the lattices whose vertices are

$$V(\mathbb{T}) = \left\{ (x, \sqrt{3}y), (x + \tfrac{1}{2}, \tfrac{\sqrt{3}}{2}(2y+1)) \ : \ x, y \in \mathbb{Z} \right\},$$

and edges that form equilateral triangles

$$(x,y) \sim (x+1, y) \qquad \text{and} \qquad (x,y) \sim (x + \tfrac{1}{2}, y + \tfrac{\sqrt{3}}{2}).$$

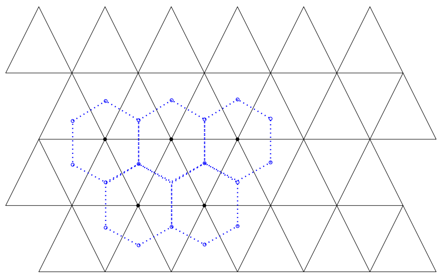
The dual lattice of $\mathbb{T}$ is the tiling of $\mathbb{R}^2$ by regular hexagons, known as the hexagonal lattice $\mathbb{H}$. So site percolation on $\mathbb{T}$ is the same as coloring the hexagons of $\mathbb{H}$ black and white randomly.

All of the results we have proved for bond percolation holds for site percolation as well: Harris' Lemma, Russo's formula, the BK inequality, Aizenman-Barsky, and in amenable graphs we have the uniqueness of the infinite component (Aizenman-Kesten-Newman or Burton-Keane).

9.5 Critical Probability in Planar Site Percolation

Let us redo Zhang's argument in the site percolation case. In fact, we will first make some general definitions. We will find it convenient to identify $\mathbb{R}^2$ with $\mathbb{C}$.

We consider percolation on the faces of $\mathbb{H}$. For Zhang's argument all we require is some shape with symmetry. For example, in $\mathbb{H}$ we may take the rhombus.



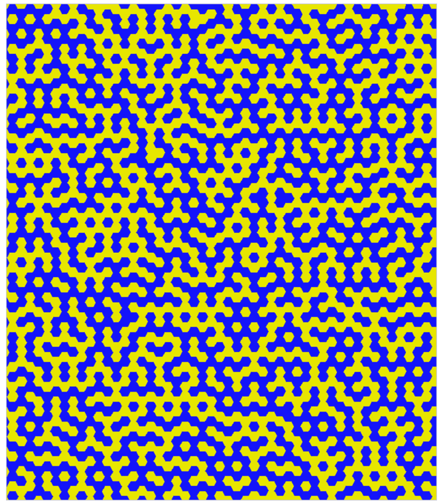


Figure 9.4: Left: The triangular lattice $\mathbb{T}$ and part of its dual lattice, the hexagonal (or honeycomb) lattice $\mathbb{H}$. Right: Percolation on the faces of $\mathbb{H}$ = site percolation on $\mathbb{T}$.

Theorem 9.5.1 For percolation on the faces of the hexagonal lattice $\mathbb{H}$ (or, equivalently, site percolation on the triangular lattice $\mathbb{T}$), $p_c = \frac{1}{2}$. Moreover, $\theta(p_c) = 0$.

Proof. We start by proving that $p_c \geq \frac{1}{2}$. Assume for a contradiction that $\theta(1/2) > 0$. Let R_n be the rhombus of hexagons of side-length n centered around the origin. Let $\partial_{\mathsf{NE}} R, \partial_{\mathsf{NW}} R, \partial_{\mathsf{SE}} R, \partial_{\mathsf{SW}} R$ be the north-east, north-west, south-east and south-west boundaries of the rhombus (corners are shared by two boundaries). Let $R^\circ = R \setminus (\partial_{\mathsf{NE}} R \cup \partial_{\mathsf{NW}} R \cup \partial_{\mathsf{SE}} R \cup \partial_{\mathsf{SW}} R)$ be the interior of R. For $\xi \in \{\mathsf{NE}, \mathsf{NW}, \mathsf{SE}, \mathsf{SW}\}$ let Let

$$A_\xi = \{\omega \ : \ \omega_{0,R^\circ} \in \{\partial_\xi R \leftrightarrow \infty\}\} .$$

This is the event that the corresponding boundary is connected to the infinite component without using sites in R°.

By flipping over the real and imaginary axis, we have that $\mathbb{P}[A_{\mathsf{NE}}] = \mathbb{P}[A_{\mathsf{NW}}] = \mathbb{P}[A_{\mathsf{SE}}] = \mathbb{P}[A_{\mathsf{SW}}]$. Moreover, these are increasing events, and $A_{\mathsf{NE}} \cup A_{\mathsf{NW}} \cup A_{\mathsf{SE}} \cup A_{\mathsf{SW}} = \{R \leftrightarrow \infty\}$. So by Harris' Lemma,

$$\mathbb{P}[R \leftrightarrow \infty] = 1 - \mathbb{P}[\cap_\xi A_\xi^c] \leq 1 - (1 - \mathbb{P}[A_{\mathsf{NE}}])^4,$$

Or

$$\mathbb{P}[A_{\mathsf{NE}}] \geq 1 - (1 - \mathbb{P}[R \leftrightarrow \infty])^{1/4}.$$

Now, by the assumption that $\theta(1/2) > 0$, taking n to infinity, the probability of $R \leftrightarrow \infty$ tends to 1, and can be made as large as we wish. Thus, if it is at least $1 - 5^{-4}$, we get that $\mathbb{P}[A_{\mathsf{NE}}] \geq \frac{4}{5}$.

Here we get to the part where $\frac{1}{2}$ is special. The above computation is valid for any $p > p_c$. But if $\theta(1/2) > 0$ (or $p_c < \frac{1}{2}$), then we get for any $\xi \in \{\mathsf{NE}, \mathsf{NW}, \mathsf{SE}, \mathsf{SW}\}$, that $\mathbb{P}[A_\xi] \geq \frac{4}{5}$ *and* $\mathbb{P}[A'_\xi] \geq \frac{4}{5}$, where $A'_\xi = \{1 - \omega \ : \ \omega \in A_\xi\}$ is the event that ∂_ξ is connect to an infinite closed component.

Thus, we conclude that for $A = A_{\mathsf{NE}} \cap A'_{\mathsf{SE}} \cap A_{\mathsf{SW}} \cap A'_{\mathsf{NW}}$,

$$\mathbb{P}[A^c] \leq \mathbb{P}[A^c_{\mathsf{NE}}] + \mathbb{P}[(A'_{\mathsf{SE}})^c] + \mathbb{P}[A^c_{\mathsf{SW}}] + \mathbb{P}[(A'_{\mathsf{NW}})^c] \leq \frac{4}{5}.$$

So $\mathbb{P}[A] \geq \frac{1}{5}$. Let E be the event that all sites in R° are open. Since A does not depend on the configuration of sites in R°, we have that A, E are independent, and

$$\mathbb{P}[A, E] \geq 2^{-|R^\circ|} \cdot \frac{1}{5} > 0.$$

But $A \cap E$ implies that there exist at least two infinite closed (red) components, on from ∂_{SE} and one from ∂_{NW} that cannot intersect each other because the infinite open (black) component connected to R separates them. This contradicts Aizenman-Kesten-Newman (uniqueness of the infinite component in the amenable graph $\mathbb{T}$).

The other direction uses the Aizenman-Barsky Theorem (exponential decay of radius of open clusters).

Assume for a contradiction that $p_c > \frac{1}{2}$. Let C be the set of corners of R. Let A be the event that there is an open (black) path in R connecting $\partial_{\mathsf{NW}} R \setminus C$ to $\partial_{\mathsf{SE}} R \setminus C$. Let B be the event that there is a closed (red) path in R connecting $\partial_{\mathsf{SW}} R \setminus C$ to $\partial_{\mathsf{NE}} R \setminus C$. We have already seen that $\mathbb{P}[A] + \mathbb{P}[B] = 1$. However, the symmetry of the rhombus, together with the fact that both red and black have the same probability gives, $\mathbb{P}_{1/2}[A] = \mathbb{P}_{1/2}[B] = \frac{1}{2}$. Most importantly, this is uniformly bounded away from 0 as $n \to \infty$.

However, for large n, by Aizenman-Barsky,

$$\mathbb{P}[A] \leq \mathbb{P}[\exists x \in \partial_{\mathsf{NW}} R \ : \ x \leftrightarrow \partial_{\mathsf{SE}} R] \leq n \, \mathbb{P}[x \leftrightarrow \partial_n(x)] \leq n e^{-cn} \to 0,$$

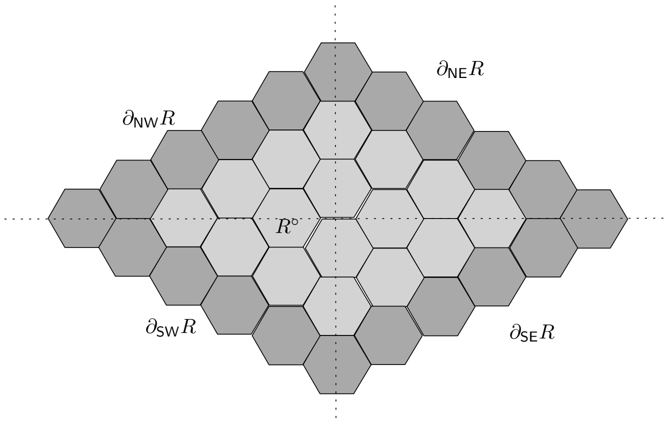


Figure 9.5: The rhombus R ($n = 6$).

a contradiction. $\square$

Chapter 10

The Cardy-Smirnov Formula

10.1 Cardy-Smirnov Formula

If $h \in \delta\mathbb{H}$ is some hexagon, we may identify it with an open set in $\mathbb{C}$ - namely the open hexagon itself. If $h \sim h'$ are adjacent hexagons in $\delta\mathbb{H}$, we identify the edge $h \sim h'$ with the boundary between h and h' in $\mathbb{C}$.

Let D be a simply connected pre-compact domain in $\mathbb{C}$ and let a, b, c, d be four points on the boundary of D, in counter-clockwise order. That is, the boundary of D is given by a Jordan curve $\gamma : [0,1] \to \mathbb{C}$, and $\gamma(t_a) = a, \gamma(t_b) = b, \gamma(t_c) = c, \gamma(t_d) = d$ for some $t_a < t_b < t_c < t_d$ in $(0,1)$. For $\xi, \zeta \in \{a, b, c, d\}$, we denote by $(\xi, \zeta)_D$ the part of γ strictly between ξ and ζ; *e.g.* $(a, b) = \gamma(t_a, t_b)$.

Given a percolation configuration $\omega \in \{0,1\}^{\mathbb{H}}$, we want to "color" $\bar{D}$ in such a way that the ω-open hexagons are black, ω-closed hexagons are red, edges between open (resp. closed) hexagons are black (resp. red), and the boundary ∂D is colored black on $(a, b) \cup (c, d)$ and red on $(b, c) \cup (d, a)$. This is done as follows.

For all $z \in \mathbb{C}$, there are three disjoint possibilities: z is in exactly on hexagon of $\delta\mathbb{H}$, z is on the boundary between two adjacent hexagons of $\delta\mathbb{H}$, or z is on the vertex where three adjacent hexagons of $\delta\mathbb{H}$ intersect.

Define a function $f_\omega : \mathbb{C} \to \{0, 1, -1\}$. (We think of $f_\omega(z) = 1$ meaning "z is black", $f_\omega(z) = 0$ meaning "z is red", and $f_\omega(z) = -1$ meaning "z is not colored" or "z is white".) For z in exactly on hexagon h set $f_\omega(z) = \omega(h)$. If z is on the boundary of two hexagons $h \sim h'$, set

$$f_\omega(z) = \begin{cases} -1 & \text{if } \omega(h) \neq \omega(h') \\ \omega(h) & \text{if } \omega(h) = \omega(h'). \end{cases}$$

Similarly, if z is on the intersection of three hexagons h, h', h'', set

$$f_\omega(z) = \begin{cases} \omega(h) & \text{if } \omega(h) = \omega(h') = \omega(h'') \\ -1 & \text{otherwise.} \end{cases}$$

This extends ω to a coloring of all $\mathbb{C}$.

Now, given the domain D with four marked points a, b, c, d on the boundary, we want the boundary to be compatible with coloring $(a, b) \cup (c, d)$ black and $(b, c) \cup (d, a)$ red. Thus, define $f_{D,\omega} : \bar{D} \to \{0, 1, -1\}$ by forcing $f_{D,\omega}$ to be 1 on hexagons that intersect (a, b) or (c, d) and forcing $f_{D,\omega}$ to be 0 on hexagons that intersect (b, c) or (d, a). If there is a conflict (points z that lie on hexagons of different colors, or hexagons that intersect different parts of the boundary of different colors) set $f_{D,\omega} = -1$.

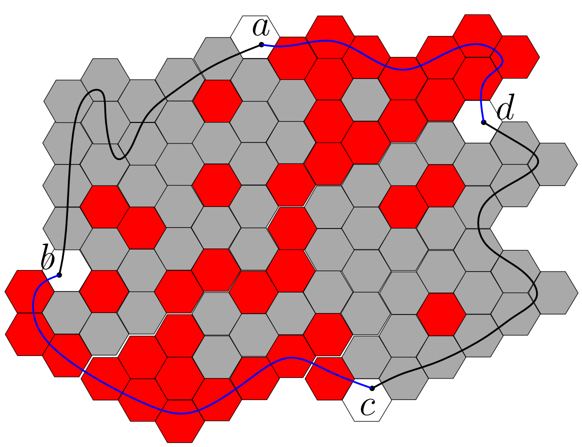


Figure 10.1: A domain D with four marked points a, b, c, d. The hexagonal lattice $\mathbb{H}$ is drawn, and is colored according to some configuration ω. Can you find the red crossing from (b, c) to (d, a)?

We can define random variables

$$B(\omega) = B_{D,a,b,c,d,\delta}(\omega) = f_{D,\omega}^{-1}(1) \qquad \text{and} \qquad R(\omega) = R_{D,a,b,c,d,\delta}(\omega) = f_{D,\omega}^{-1}(0).$$

These are the black and red components in $\bar{D}$. One may then verify that because we are working in the plane, and because D is simply connected, either (a, b) is connected to (c, d) by a black path, or (b, c) is connected to (d, a) by a red path. These events are disjoint, since there cannot be both red and black paths intersecting in D.

We can now ask what is the probability that there exists a black path connecting (a, b) to (c, d)? Let us denote this probability by $P(D, a, b, c, d, \delta)$.

An amazing conjecture by Aizenman, Langlands, Pouliot and Saint-Aubin states that the limit $P(D, a, b, c, d) = \lim_{\delta\to 0} P(D, a, b, c, d, \delta)$ should exist and should be conformal invariant; that is, if $\phi : D \to D'$ is a conformal map then

$$P(\phi(D), \phi(a), \phi(b), \phi(c), \phi(d)) = P(D, a, b, c, d).$$

John Cardy

Cardy later predicted the exact formula for this probability, which, as Lennart Carleson observed, is simplest stated on the equilateral triangle: For $T =$ the equilateral triangle, and a, b, c the vertices of T, suppose that $[c, a] = [0, 1]$ on the real axis. Then for any $d \in (0, 1)$, $P(T, a, b, c, d) = d$. (A miracle!)

Stanislav Smirnov

Cardy's (non-rigorous) arguments, stemming from conformal field theory, are valid for any "nice" planar lattice. In one of the beautiful works of the field, Smirnov proved Cardy's formula and the conformal invariance for the triangular lattice $\mathbb{T}$.

Theorem 10.1.1 **(Cardy-Smirnov Formula)** Consider site percolation on the triangular lattice $\mathbb{T}$. Then, the limit $P(D, a, b, c, d)$ exists and is conformal invariant in the sense

$$P(\phi(D), \phi(a), \phi(b), \phi(c), \phi(d)) = P(D, a, b, c, d)$$

for any conformal map ϕ.

Moreover, for an equilateral triangle T of unit side length with vertices a, b, c, and any point d on (c, a), the formula for this probability is given by

$$P(T, a, b, c, d) = |d - c|.$$

10.2 Color Switching

We will prove the main steps in Smirnov's fabulous proof. For convenience, and simplicity of the presentation, we will restrict to *nice domains*.

Let $\delta > 0$. A **nice domain** D at scale δ, is a union of hexagons in $\delta\mathbb{H}$, such that D is simply connected, and such that the boundary of D forms a simple path in $\delta\mathbb{H}$.

We also want to consider points in the complex plane that lie on the intersections of the hexagons. Let $\mathbb{C}_\delta$ be the set of vertices of the hexagonal lattice $\delta\mathbb{H}$ (that is points on the intersection of three hexagons).

Let $\partial_\delta D$ be the points in $\mathbb{C}_\delta$ that lie on two hexagons in the boundary ∂D. Each such point has a unique adjacent edge that is an edge between hexagons. If we mark k such points on the boundary of D, then this partitions the boundary of D into k disjoint parts.

Let us restate the planar duality lemma that we have already used many times.

Lemma 10.2.1 Let D be a nice domain with 4 marked points $a, b, c, d \in \partial_\delta D$. For $x, y \in \{a, b, c, d\}$ let (x, y) be the part of the boundary between points x and y. Let A be the event that there is an open (black) crossing from (a, b) to (c, d), and let B be the event that there is a closed (red) crossing from (b, c) to (d, a). Let Ω be the set of percolation configurations $\omega \in \{0,1\}^D$ with boundary values $\omega\big|_{(a,b)\cup(c,d)} = 1, \omega\big|_{(b,c)\cup(d,a)} = 0$. Then,

$$\Omega = A \biguplus B.$$

That is, if we force the boundary to be black on $(a, b) \cup (c, d)$ and red on $(b, c) \cup (d, a)$, then there is either a black crossing between (a, b) and (c, d) or a red crossing between (b, c) and (d, a), but not both.

Proof. Start an exploration on the edges in-between hexagons from the unique edge adjacent to point b, keeping red on the right and black on the left. When reaching a fresh hexagon, turn left if it is red and right if it is black. This defines a unique path from b into D that can only exit D at an edge entering a vertex in $\partial_\delta D$, which is adjacent to hexagons of different colors. The only such edges (under our boundary conditions) are those entering a, c, d. But exiting at d would imply that red is on the left and black on the right, which is the wrong orientation. So the exploration path must exit at either a or c.

If the exploration path exits at a - the red hexagons to the right of the path form a red connection from (b, c) to (d, a). If the exploration path exits at c the black hexagons to the left of the path form a black connection from (a, b) to (d, a).

Two such connections cannot exist, since this would give two disjoint paths in D, one from (a, b) to (c, d) and one from (b, c) to (d, a) that do not intersect. This contradicts the fact that D is simply connected. $\square$

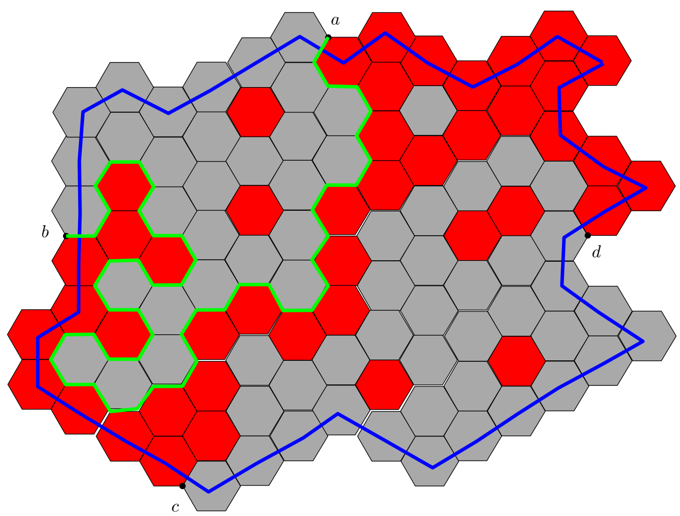


Figure 10.2: A nice domain D. The path tracing the boundary is marked in blue. Four points in $\mathbb{C}_\delta$ dividing the boundary into four parts are shown. The exploration path from b is marked in green.

Let D be a nice domain, $x \in D$ and $A \subset \partial D$. We write $x \leftrightarrow_B A$ for the event that x is connected to A by a black (open) path in D. Similarly, we write $x \leftrightarrow_R A$ for the event that x is connected to A by a red (closed) path in D. Note that x must be open / black (resp. closed / red) if $x \leftrightarrow_B A$ (resp. $x \leftrightarrow_R A$). So these events are disjoint.

The following lemma is a subtle step in Smirnov's proof, and is usually dubbed the *Color Switching Lemma*.

Lemma 10.2.2 **(Color Switching)** Let D be a nice domain of scale δ. Let $a, b, c \in \partial_\delta D$ and let $A_1 = (a,b), A_2 = (b,c), A_3 = (c,a)$. Let x_1, x_2, x_3 be three adjacent hexagons in $D \setminus \partial D$ (so their centers form a triangle).

For $j \in \{1,2,3\}$ write

$$B_j = \{x_j \leftrightarrow_B A_j\} \qquad \text{and} \qquad R_j = \{x_j \leftrightarrow_R A_j\}\,.$$

Then,

$$\mathbb{P}_{\frac{1}{2}}[B_1 \circ B_2 \circ R_3] = \mathbb{P}_{\frac{1}{2}}[B_1 \circ R_2 \circ B_3] = \mathbb{P}_{\frac{1}{2}}[R_1 \circ B_2 \circ B_3].$$

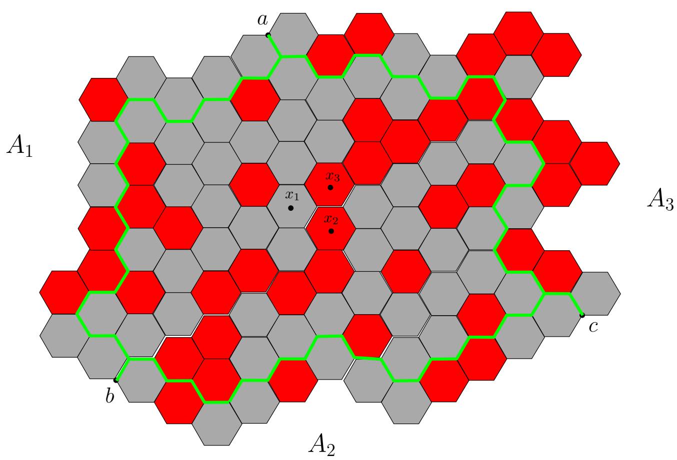


Figure 10.3: A nice domain with three marked points on the boundary, and three adjacent hexagons x_1, x_2, x_3 inside. Note the event $B_1 \circ R_2 \circ R_3$.

Remark 10.2.3 It is important not to mix this up with the fact that any color combination is possible. The Color Switching Lemma says nothing about three disjoint paths of the *same* color (*i.e.* $\mathbb{P}[B_1 \circ B_2 \circ B_3]$).

However, note that since we are at $p = \frac{1}{2}$, we may switch all black to red and vice-versa, so $\mathbb{P}_{\frac{1}{2}}[B_1 \circ B_2 \circ R_3] = \mathbb{P}_{\frac{1}{2}}[R_1 \circ R_2 \circ B_3]$, and similarly for other combinations.

Proof. The first step is to note that it suffices to prove

$$\mathbb{P}[B_1 \circ R_2 \circ B_3] = \mathbb{P}[B_1 \circ R_2 \circ R_3]. \tag{10.1}$$

Indeed, given (10.1), switching black to red and vice-versa (recall that we are at $p = \frac{1}{2}$),

$$\mathbb{P}_{\frac{1}{2}}[B_1 \circ R_2 \circ B_3] = \mathbb{P}_{\frac{1}{2}}[B_1 \circ R_2 \circ R_3] = \mathbb{P}_{\frac{1}{2}}[R_1 \circ B_2 \circ B_3],$$

which is the second equality in the lemma.

Relabeling $1 \mapsto 2 \mapsto 3 \mapsto 1$, we also have

$$\mathbb{P}[B_1 \circ B_2 \circ R_3] = \mathbb{P}[R_3 \circ B_1 \circ B_2] = \mathbb{P}[B_3 \circ R_1 \circ B_2] = \mathbb{P}[R_1 \circ B_2 \circ B_3].$$

We proceed with the proof of (10.1). Note that (10.1) is equivalent to

$$\mathbb{P}[B_1 \circ R_2 \circ B_3 \mid B_1 \cap R_2] = \mathbb{P}[B_1 \circ R_2 \circ R_3 \mid B_1 \cap R_2]. \tag{10.2}$$

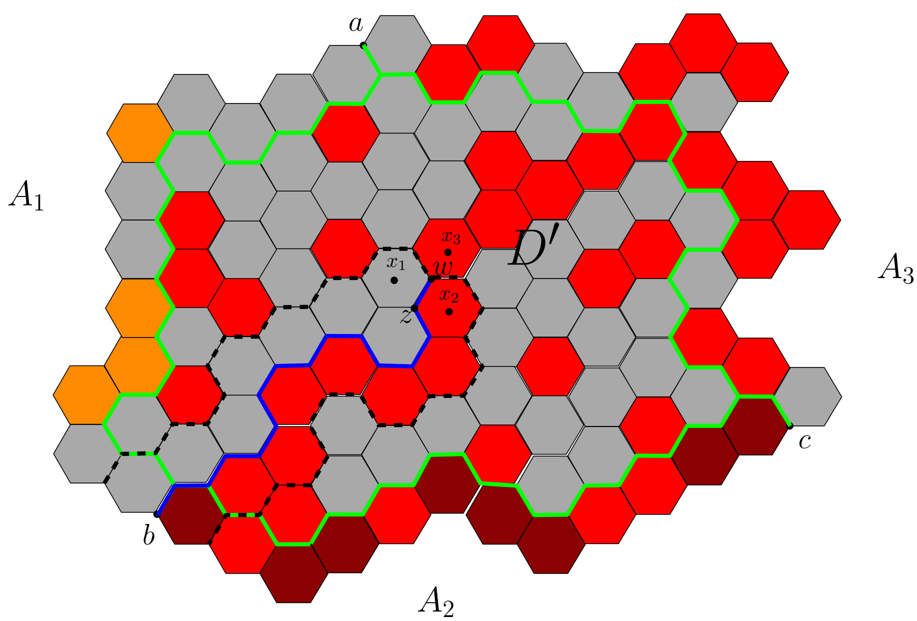


Figure 10.4: The proof of (10.2). The blue line is the interface γ, which only explores those hexagons in the dotted black boundary. What is left over is D'.

Suppose we are on the event $B_1 \cap R_2$. Start an exploration at b into the edge between A_1 and A_2. If we think of A_1 as being black and A_2 as red, we may explore the interface until we reach the edge (z, w) between the hexagons x_1, x_2, such that w is at the intersection of the three hexagons x_1, x_2, x_3.

This interface is explored so that black is always on the left, red is on the right, and A_1 is considered black, A_2 is considered red.

Now, let γ be the path explored, and let D' be the domain D with all hexagons adjacent to γ removed. Note that the hexagons adjacent to γ include a black path from x_1 to A_1 and a red path from x_2 to A_2. Also note that $x_3 \in D'$.

The main observation is that due to planarity, conditioned on γ being the interface stemming from the event $B_1 \cap R_2$, the event $B_1 \circ R_2 \circ B_3$ holds if and only if there exists a black path in D' from x_3 to A_3. Similarly, conditioned on γ, $B_1 \circ R_2 \circ R_3$ holds if and only if there exists a red path in D' from x_3 to A_3.

Recalling that we are at $p = \frac{1}{2}$, and that all hexagons in D' are independent of those adjacent to γ, we get

$$\begin{aligned}\mathbb{P}[B_1 \circ R_2 \circ B_3 \mid \gamma] &= \mathbb{P}_{\frac{1}{2}}[\exists \text{ black } \alpha : x_3 \to A_3 \subset D'] = \mathbb{P}_{\frac{1}{2}}[\exists \text{ red } \alpha : x_3 \to A_3 \subset D'] \\ &= \mathbb{P}[B_1 \circ R_2 \circ R_3 \mid \gamma].\end{aligned}$$

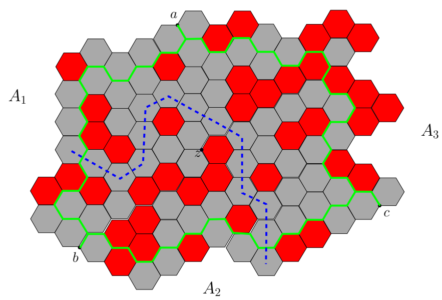
Summing over all possible γ, since the event $B_1 \cap R_2$ is the union of such Interfaces, we obtain (10.2). $\square$

10.3 Harmonic Functions

In this section we will consider special functions on the vertices of the hexagonal lattice $\delta\mathbb{H}$; *i.e.* functions on $\mathbb{C}_\delta$.

Throughout this section fix a nice domain D, with three points a, b, c on $\partial_\delta D$, separating ∂D into $A_1 = (a, b), A_2 = (b, c), A_3 = (c, a)$. Let $z \in \mathbb{C}_\delta$ be some $\delta\mathbb{H}$-vertex in the interior of D. Define the event $S^3(z) = S^3_\delta(z)$ to be the event that there exists a *simple* open (black) path from A_1 to A_2 that separates z from A_3. We stress that this path must be simple, that is, no hexagon may be visited more than once. The events $S^1(z), S^2(z)$ are defined similarly.

Lemma 10.3.1 Let $D, a, b, c, x_j, A_j, B_j, R_j$ be as in the Color Switching Lemma (Lemma 10.2.2). Assume that x_1, x_2, x_3 are in counter-clockwise order, and also A_1, A_2, A_3 (or, equivalently, a, b, c). Let (z, w) be the edge in the hexagonal lattice $\delta\mathbb{H}$ that is in between the hexagons x_1, x_2, such



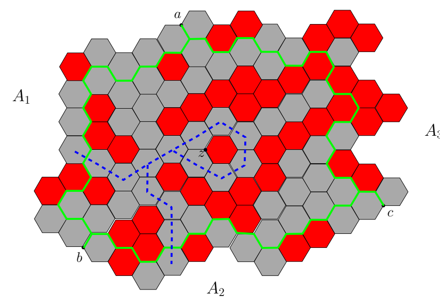


Figure 10.5: Left: The event $S^3(z)$. Right: $S^3(z)$ does not hold, because the path is not simple.

that w is at the intersection of all three hexagons x_1, x_2, x_3.

Then, we have the equality of events:

$$S^3(z) \setminus S^3(w) = B_1 \circ B_2 \circ R_3.$$

Proof. The direction $B_1 \circ B_2 \circ R_3 \subset S(z) \setminus S(w)$ is simpler. Indeed, if $B_1 \circ B_2 \circ R_3$ holds, then there are 3 disjoint paths, $\gamma_1 : A_1 \to x_1, \gamma_2 : x_2 \to A_2, \gamma_3 : x_3 \to A_3$ such that γ_1, γ_2 are black and γ_3 is red. The composed path $\gamma_2\gamma_1$ is a black path from A_1 to A_2 that passes through $x_1 \mapsto x_2$. Since (z, w) is the edge between x_1 and x_2, this path must separate z from w. Since there is a closed path from w to A_3, it must be that $\gamma_2\gamma_1$ separates z from A_3, but does not separate w from A_3.

For the other direction, assume that $S(z) \setminus S(w)$ holds. Thus, there exists a black (open) path $\gamma : A_1 \to A_2$ separating z from A_3. Suppose that $|\gamma| = n$. Let β be the path in D that starts on $\gamma_n \in A_2$, continues in A_2 in clockwise direction, until reaching A_1, continues on A_1 in clockwise direction and stops once reaching γ_0. The path $\beta\gamma$ is a simple cycle that surrounds z but not w (otherwise $S^3(z) \setminus S^3(w)$ would not hold). So γ must pass through the edge (z, w), meaning that $x_1, x_2 \in \gamma$ but $x_3 \notin \gamma$. Because γ goes from A_1 to A_2, and x_1, x_2, x_3 are in cyclic order, this implies that γ is composed of two disjoint parts: a black path from A_1 to x_1 and another disjoint black path from x_2 to A_2.

Now, let D' be the domain that is the component of $D \setminus \gamma$ containing x_3. If D' does not contain part of A_3 then w is separated by γ from A_3, which contradicts $S^3(z) \setminus S^3(w)$. Let $D'' = D' \cup \gamma$. Suppose that α_1 is the part

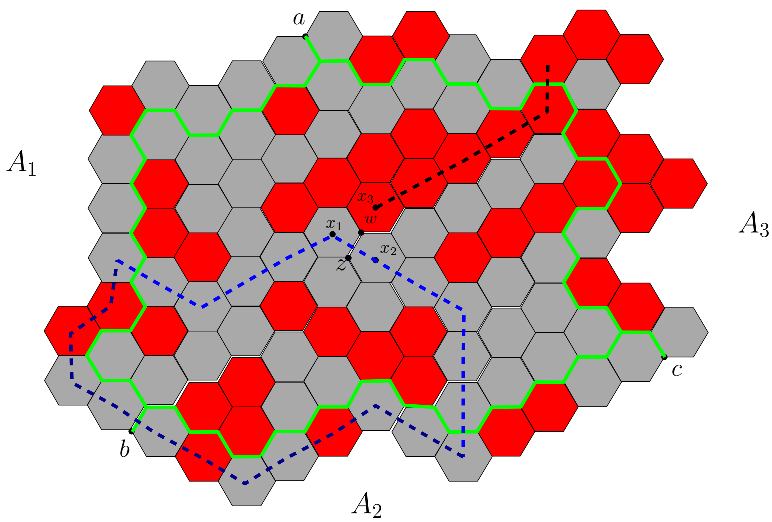


Figure 10.6: Proof of Lemma 10.3.1.

of γ from A_1 to x_1 and α_2 is the part of γ from x_2 to A_2. Let $A_1' = (A_1 \cap D'') \cup \alpha_1 \setminus \{x_1\}$ and $A_2' = (A_2 \cap D'') \cup \alpha_2 \setminus \{x_2\}$. Let $A_3' = A_3 \cap D''$ and $A_4' = \{x_1, x_2, x_3\}$. These sets decompose the boundary of D'' into four parts. Our duality arguments give that either A_1' is connected to A_2' by a black path in D'', or A_3' is connected to A_4' by a red path in D''. In the first case, this path joined with γ would separate w from A_3 in D, contradicting $S^3(z) \setminus S^3(w)$. So there must exist a red path connecting A_3' to A_4'.

If x_3 was black, then w would be separated from A_3 in D. So , since x_1, x_2 are black, we have that x_3 is connected to A_3 in D by a red path, which is of course disjoint from the disjoint black paths $\alpha_1 : A_1 \to x_1$ and $\alpha_2 : x_2 \to A_2$. We thus conclude that $B_1 \circ B_2 \circ R_3$ holds. □

10.3.1 Contour Integrals

Recall that the hexagonal lattice $\delta\mathbb{H}$ is dual to the triangular lattice $\delta\mathbb{T}$. That is, if (x, y) is a directed edge in $\delta\mathbb{T}$, then for $\rho = \frac{i}{\sqrt{3}}$, the dual directed edge in $\delta\mathbb{H}$ is (z, w) where

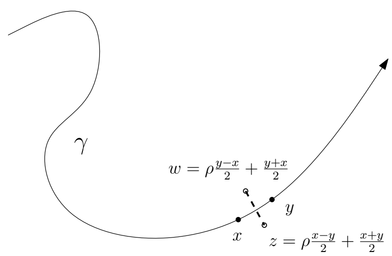
$$w = \rho\frac{y-x}{2} + \frac{y+x}{2} \qquad \text{and} \qquad z = \rho\frac{x-y}{2} + \frac{x+y}{2}$$

(multiplication by i rotates by $\frac{\pi}{2}$ and $\sqrt{3}$ is the scale. There is of course a choice here to rotate either clockwise or counter-clockwise, which we have arbitrarily taken to be counter-clockwise.) Recall that vertices of $\delta\mathbb{T}$ are the centers of the hexagons in $\delta\mathbb{H}$. For a directed edge (x, y) in $\delta\mathbb{T}$, we write $[x, y]$ for the corresponding dual directed edge in $\delta\mathbb{H}$. If (z, w) is a directed edge in $\delta\mathbb{H}$ we write $[z, w]$ for the corresponding primal edge in $\delta\mathbb{T}$. Note that for $z \sim w$ in $\delta\mathbb{H}$, if $[z, w] = (x, y)$ then $\rho(y - x) = (w - z)$.

Let γ be a simple path in $\delta\mathbb{T} \cap D$ Define the contour integral along γ of a function $\phi = \phi_\delta$ on $\mathbb{C}_\delta \cap D$ by

$$\oint_\gamma \phi = \sum_{j=1}^{|\gamma|} (\gamma_j - \gamma_{j-1})\phi([\gamma_{j-1}, \gamma_j]),$$

where $\phi([x, y]) = \phi([x, y]^+) = \phi(\rho\frac{y-x}{2} + \frac{y+x}{2})$. If $\phi = \phi_\delta$ converges uniformly to some limit ψ, this integral will converge to the usual complex contour integral of ψ.



Note that if γ is a simple closed path in $\delta\mathbb{T} \cap D$ (so $\gamma_0 = \gamma_{|\gamma|}$ is the only self intersection point), then it separates the complex plane $\mathbb{C}$ into two components, an interior, denoted $\mathrm{Int}(\gamma)$ and exterior, which is the component containing ∞ (Jordan's theorem). If γ is oriented counter-clockwise, then

$$\oint_\gamma \phi = \sum_{\mathrm{Int}(\gamma) \ni w \sim z \notin \mathrm{Int}(\gamma)} \rho^{-1}(w - z)\phi(w).$$

One more definition, which is to avoid technical complications: If D is some nice domain of scale δ, and $a, b, c \in \partial_\delta D$, for $z \in D$ define $\mathrm{rad}_{D,a,b,c}(z) = \max_{j=1,2,3} \mathrm{dist}(z, A_j)$, where as usual $A_1 = (a, b), A_2 = (b, c), A_3 = (c, a)$. The radius of D is then defined to be

$$\mathrm{rad}(D) = \inf_{z \in D} \mathrm{rad}_{D,a,b,c}(z).$$

✓ For a point $z \in \mathbb{C}_\delta \cap D$ define $H_j(z) = \mathbb{P}[S^j(z)]$. For a directed edge in the hexagonal lattice, $z \sim w \in \mathbb{C}_\delta \cap D$, define $P_j(z,w) = \mathbb{P}[S^j(z) \setminus S^j(w)]$. Note that

$$H_j(z) - H_j(w) = P_j(z,w) - P_j(w,z).$$

Set $\tau = e^{i\frac{2\pi}{3}}$. Define

$$H = H_\delta = \tau H_1 + \tau^2 H_2 + \tau^3 H_3 \qquad \text{and} \qquad F = F_\delta = H_1 + H_2 + H_3.$$

Lemma 10.3.2 For any simple closed path γ in $\delta\mathbb{T} \cap D$, of Euclidean length L,

$$\Big| \oint_\gamma H \Big| \le 3L\,\mathbb{P}[x \leftrightarrow \partial_n(x)] \qquad \text{and} \qquad \Big| \oint_\gamma F \Big| \le 3L\,\mathbb{P}[x \leftrightarrow \partial_n(x)],$$

where $n = \lceil \delta^{-1}\mathrm{rad}(D) \rceil$.

Proof. We can without loss of generality assume that γ is oriented counter-clockwise. Set $I = \mathrm{Int}(\gamma) \cap \mathbb{C}_\delta$.

For $w \in \mathbb{C}_\delta \cap D$, consider the sum

$$\sum_{z \sim w} (w - z)\phi(w) = (1 + \tau + \tau^2)(w - z')\phi(w) = 0,$$

for some fixed $z' \sim w$. Thus,

$$\begin{aligned} 0 &= \sum_{w \in I, z \sim w} (w - z)\phi(w) = \sum_{I \ni w \sim z \in I} (w - z)\phi(w) + \rho \oint_\gamma \phi \\ &= \frac{1}{2} \sum_{I \ni w \sim z \in I} (w - z)(\phi(w) - \phi(z)) + \rho \oint_\gamma \phi. \end{aligned}$$

Taking $\phi = H_j$, we obtain,

$$\begin{aligned} \rho \oint_\gamma H_j &= \frac{1}{2} \sum_{I \ni w \sim z \in I} (z - w)(P_j(w,z) - P_j(z,w)) = \sum_{I \ni w \sim z \in I} (w - z)P_j(z,w) \\ &= \sum_{w \in I, z \sim w} (w - z)P_j(z,w) + \sum_{I \ni w \sim z \notin I} (z - w)P_j(z,w). \end{aligned}$$

Now, fix $w \in I$, and let z_1, z_2, z_3 be the neighbors of w in counter-clockwise order. Let $P_{j,k} = P_j(z_k, w)$. The Color Switching Lemma together with Lemma 10.3.1 give that

$$P_{j,k} = P_{j+1,k+1} = P_{j+2,k+2}. \tag{10.3}$$

(Throughout, we always take indices modulo 3.) Thus,

$$\sum_{z\sim w}(w-z)\sum_{j=1}^{3}P_j(z,w)=(w-z_3)\sum_{j,k=1}^{3}\tau^k P_{j,k}=(1+\tau+\tau^2)\sum_{k=1}^{3}P_{1,k}=0,$$

and also

$$\sum_{z\sim w}(w-z)\sum_{j=1}^{3}\tau^j P_j(z,w)=(w-z_3)\sum_{j,k=1}^{3}\tau^{k+j}P_{j,k}=(1+\tau+\tau^2)\sum_{k=1}^{3}P_{1,k}=0.$$

We conclude that

$$\rho\oint_\gamma H=\sum_{I\ni w\sim z\notin I}(z-w)(\tau P_1(z,w)+\tau^2 P_2(z,w)+\tau^3 P_3(z,w)),$$

and

$$\rho\oint_\gamma F=\sum_{I\ni w\sim z\notin I}(z-w)(P_1(z,w)+P_2(z,w)+P_3(z,w)).$$

Finally, note that the event $S^j(z)\setminus S^j(w)$ implies that for every hexagon that z is on, there exists a mono-chromatic path from that hexagon to the corresponding part of the boundary. This is just Lemma 10.3.1. Since one of these boundary parts is at distance at least $\mathrm{rad}(D)$ from z, and since the distance between two adjacent hexagon centers is δ, we get that for $n=\lceil\delta^{-1}\mathrm{rad}(D)\rceil$,

$$\mathbb{P}[S^j(z)\setminus S^j(w)]\le\mathbb{P}[x\leftrightarrow\partial_n(x)].$$

Thus,

$$\begin{aligned}\Big|\oint_\gamma F\Big|&=\Big|\sum_{j=1}^{|\gamma|}(\gamma_{j-1}-\gamma_j)(P_1+P_2+P_3)([\gamma_{j-1},\gamma_j]^-,[\gamma_{j-1},\gamma_j]^+)\Big|\\&\le\delta|\gamma|3\,\mathbb{P}[x\leftrightarrow\partial_n(x)],\end{aligned}$$

and similarly,

$$\Big|\oint_\gamma H\Big|\le\delta|\gamma|\cdot 3\,\mathbb{P}[x\leftrightarrow\partial_n(x)].$$

Since $\delta|\gamma|$ is the Euclidean length of γ, this completes the proof. □

10.3.2 Proof of the Cardy-Smirnov Formula

✓ Assume that the domain D is a discrete approximation of some simply connected domain $\mathcal{D}$. We will **assume** (but not prove) that the functions H_j converge uniformly to limiting functions h_1, h_2, h_3 on $\mathcal{D}$ as $\delta \to 0$. (This can be shown via the Arzelà-Ascoli Theorem, with a proper Hölder continuity estimate on H_j.) Given this convergence, we have limiting functions $h = \lim H_\delta = \tau h_1 + \tau^2 h_2 + \tau^3 h_3$ and $f = \lim F_\delta = h_1 + h_2 + h_3$. Our goal is to show that h, f are harmonic, so must be uniquely defined by their boundary values. We will then use the boundary values to determine these functions.

Giacinto Morera (1856–1909)

Lemma 10.3.2 tells us that for any triangular closed simple contour γ inside $\mathcal{D}$, the contour integrals satisfy $\oint_\gamma h(z)dz = \oint_\gamma f(z)dz = 0$. This is because of uniform convergence, so the discrete contour integrals converge to their continuous counterparts, and also because as $\delta \to 0$, for $n = \lceil \delta^{-1}\mathrm{rad}(D) \rceil \to \infty$, so $\mathbb{P}[x \leftrightarrow \partial_n(x)] \to 0$, by Zhang's argument.

A famous theorem by Morera now tells us that h and f are holomorphic in $\mathcal{D}$. So h and f are determined by their boundary values.

Let us see what these boundary values are.

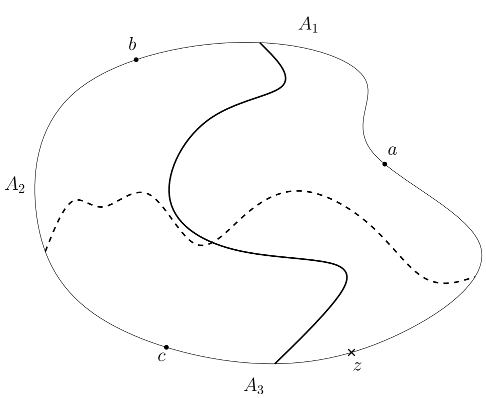


Figure 10.7: The possibilities when $z \in A_3$, which cannot occur together.

Note that for any $z \in A_3$, one cannot separate z from A_3, so $h_3(z) = 0$. Also, as in Figure 10.7, consider the four boundary parts $(a, b), (b, c), (c, z), (z, a)$. Then, either (a, b) is connected to (c, z) by a black path or (b, c) is connected to (z, a) by a red path, and exactly one of these events must hold. Since red and black have the same probability (recall, we are at $p = \frac{1}{2}$!) we get that $\mathbb{P}[S^1(z)] + \mathbb{P}[S^2(z)] = 1$, because the black and red paths above separate z from the corresponding boundaries. Taking the $\delta \to 0$ limit gives that for any $z \in A_3$,

$$h_3(z) = 0 \qquad \text{and} \qquad h_1(z) + h_2(z) = 1.$$

Of course this argument can be repeated for any $j \in \{1, 2, 3\}$, so that if $z \in A_j$ then

$$h_j(z) = 0 \qquad \text{and} \qquad h_{j+1}(z) + h_{j+2}(z) = 1.$$

What is the conclusion?

✓ First of all, for any $z \in A_j$ we get

$$f(z) = h_1(z) + h_2(z) + h_3(z) = 1.$$

So f has constant 1 boundary values, and thus must be the constant function 1.

✓ Second, let T be the equilateral triangle with vertices at $1, \tau, \tau^2$. Let $\varphi : T \to \mathcal{D}$ be the conformal map (guarantied by the Reimann mapping theorem) that maps $\varphi(\tau) = c, \varphi(\tau^2) = a, \varphi(1) = b$. Consider the holomorphic function, $g = h \circ \varphi$.

Because $h_1 + h_2 + h_3 = 1$, we have that any $z \in T$ is mapped by g to a convex combination of $\tau, \tau^2, \tau^3 = 1$, that is $g : T \to T$. Moreover, if $z \in (\tau, \tau^2)$ then $\varphi(z) \in (c, a)$ and so $h_3(\varphi(z)) = 0$ and

$$g(z) = \tau h_1(\varphi(z)) + \tau^2 h_2(\varphi(z)) \in (\tau, \tau^2).$$

Similarly, we get that $g : T \to T$ is a holomorphic map, mapping the boundary parts $(1, \tau), (\tau, \tau^2)$ and $(\tau^2, 1)$ to themselves. By standard complex analysis there is only one such function: the identity. So $h : \mathcal{D} \to T$ is the Reimann map mapping $a \mapsto \tau^2, b \mapsto 1, c \mapsto \tau$.

Finally, we recover the Cardy-Smirnov formula: For a point $d \in (c, a)$, the probability that (a, b) is connected to (c, d) by a black path is the probability that d is separated from (b, c) by a black path from (a, b) to (c, a). Thus, as $\delta \to 0$ this converges to $h_2(d)$. We have seen that

$$\phi(d) = \tau h_1(d) + \tau^2 h_2(d) = \tau + \tau(\tau - 1) h_2(d)$$

where ϕ is the unique conformal map from $\mathcal{D}$ to T mapping $a \mapsto \tau^2, b \mapsto 1, c \mapsto \tau$.

This already proves the conformal invariance.

It also gives the value of $h_2(d)$, since we can evaluate it for a specific domain, namely T itself. If $a = \tau^2, b = 1, c = \tau$ and if $d \in (c, a)$ then write $d = (1-\varepsilon)c + \varepsilon a$ where $|d - c| = |\tau^2 - \tau| \cdot \varepsilon$. Note that ϕ is the identity map, so

$$h_2(d) = \frac{d - \tau}{\tau(\tau - 1)} = \frac{d - c}{\tau^2 - \tau}.$$

$h_2(d)$ is a non-negative real number so $h_2(d) = \frac{|d-c|}{|\tau^2-\tau|} = \varepsilon$, which is exactly the formula, since $|\tau^2 - \tau| = |\tau - 1|$ is exactly the side length of the triangle.

Chapter 11

Percolation Beyond $\mathbb{Z}^d$

11.1 The Mass Transport Principle

11.1.1 A review of group actions

Recall that a group Γ is said to act on a set X if there is an **action** $\Gamma \times X \to X$, denoted $(g,x) \mapsto g.x$ such that $gh.x = g.h.x$ for all $g,h \in \Gamma, x \in X$ (associativity) and $1_\Gamma.x = x$ for all $x \in X$.

Exercise 11.1 Show that any group acts on itself by left multiplication.

Show that any group acts on itself by conjugation. $\diamond\diamond\diamond$

Exercise 11.2 Let $\Gamma = \mathrm{Aut}(G)$ be the set of automorphisms of a graph G. Show that Γ acts on G. $\diamond\diamond\diamond$

A group Γ acts **transitively** on X if for every $x,y \in X$ there exists $g \in \Gamma$ such that $g.x = y$. Thus, a transitive graph G is one that its automorphism group acts on it transitively.

If Γ acts on X, for any $x \in X$ we can define a subgroup $\Gamma_x := \{g \in \Gamma \ : \ g.x = x\}$. This is called the **stabilizer** of x. We can also define $\Gamma x = \{g.x \ : \ g \in \Gamma\}$. This is the **orbit** of x.

So a transitive action is the same as $\Gamma x = X$ for all $x \in X$.

Exercise 11.3 Show that the stabilizer of x is a subgroup.

Show that $|\Gamma x| = [\Gamma : \Gamma_x]$. ◇◇◇

One may also consider the orbit of a subset of Γ: If $S \subset \Gamma$ let $Sx = \{s.x \ : \ s \in S\}$.

Let Γ act on X. A function $f : X \times X \to \mathbb{R}$ is said to be **invariant** under Γ, if it is invariant under the diagonal action of Γ; that is, if for any $x, y \in X$ and any $g \in \Gamma$, we have $f(gx, gy) = f(x, y)$.

11.1.2 Mass Transport

Let Γ act on a graph X by automorphisms (*i.e.* $\Gamma \leq \mathrm{Aut}(X)$). Consider $\Gamma_x y = \{g.y \ : \ g.x = x\}$. Note that for any $g \in \Gamma_x$ we have that $\mathrm{dist}(g.y, x) = \mathrm{dist}(y, x)$. Thus, $\Gamma_x y \subset \{z \ : \ \mathrm{dist}(z, x) = \mathrm{dist}(y, x)\}$, which is finite.

Theorem 11.1.1 **(General Mass Transport Principle)** Let Γ be a group acting by automorphisms on a graph X. Let $f : X \times X \to [0, \infty]$ be a non-negative invariant function. Then, for any $a, b \in X$

$$\sum_{x \in \Gamma b} f(a, x) = \sum_{y \in \Gamma a} f(y, b) \cdot \frac{|\Gamma_y b|}{|\Gamma_b y|}.$$

Before proving the general mass transport principle, let us see some of the consequences: First of all, if the group acts transitively, then the sums above are just over the whole space X.

Corollary 11.1.2 If Γ acts transitively by automorphisms on a graph X and $f : X \times X \to [0, \infty]$ is a non-negative invariant function, then for any $o \in X$,

$$\sum_{x} f(o, x) = \sum_{x} f(x, o) \cdot \frac{|\Gamma_x o|}{|\Gamma_o x|}.$$

A most important concept in this context is *unimodularity*:

Definition 11.1.3 Let Γ act by automorphisms on a graph X. The action is called **unimodular** if for any $y \in \Gamma x$, we have $|\Gamma_x y| = |\Gamma_y x|$.

A graph G is called unimodular if $\mathrm{Aut}(G)$ is a unimodular action.

Exercise 11.4 Let Γ be a group acting on X. Show that for all $x, y \in X$,

$$|\Gamma_x y| = [\Gamma_x : \Gamma_x \cap \Gamma_y].$$

Show that if $|\Gamma_x| < \infty$ for all $x \in X$, then the action is unimodular. ⋄⋄⋄

The mass transport principle is in fact equivalent to unimodularity for transitive actions.

Theorem 11.1.4 **(Mass Transport Principle)** Let Γ be a transitive action of automorphisms on a graph X. Then, the action is unimodular if and only if for every $f : X \times X \to [0, \infty]$ non-negative invariant function, and every $o \in X$,

$$\sum_x f(o, x) = \sum_x f(x, o).$$

First, some algebra:

Proposition 11.1.5 Let Γ act on X. Let $\Gamma_{x,y} = \{g \in \Gamma \ : \ g.x = y\}$.

For any $x, y \in X$ we have that for any $g \in \Gamma_{x,y}$,

$$\Gamma_{x,y} = g\Gamma_x = \Gamma_y g.$$

Moreover, for any $x, y, z \in X$, if $\Gamma_{x,y} \neq \emptyset$ then $|\Gamma_{x,y} z| = |\Gamma_x z|$.

Proof. Choose $g \in \Gamma_{x,y}$. Then, if $h \in \Gamma_{x,y}$ we have that $h = gg^{-1}h$ and $g^{-1}h \in \Gamma_x$. On the other hand, if $h \in \Gamma_x$ then $gh \in \Gamma_{x,y}$. Thus, $\Gamma_{x,y} = g\Gamma_x$.

Similarly, if $h \in \Gamma_{x,y}$ then $hg^{-1} \in \Gamma_y$ and if $h \in \Gamma_y$ then $hg \in \Gamma_{x,y}$. So $\Gamma_{x,y} = \Gamma_y g$.

Now, by the first identity $\Gamma_{x,y} z = g\Gamma_x z$. The map $h.z \mapsto gh.z$ between $\Gamma_x z$ and $g\Gamma_x z$ is a bijection; indeed, it is injective by associativity, and it is surjective by definition. So

$$|\Gamma_{x,y} z| = |g\Gamma_x z| = |\Gamma_x z|.$$

$\square$

Proof of Theorem 11.1.4. Let Γ be a transitive action. If Γ is unimodular, then the theorem follows directly by the mass transport principle, since $|\Gamma_x o| = |\Gamma_o x|$ for all o, x by the definition of unimodularity.

Now assume that Γ is transitive and $\sum_x f(o,x) = \sum_x f(x,o)$ for every non-negative invariant function as in the theorem. Fix $a, b \in X$ and set $f(x,y) = \mathbf{1}_{\{y \in \Gamma_{a,x} b\}}$ where $\Gamma_{a,x} = \{g \in \Gamma \ : \ g.a = x\}$. f is invariant since $f(g.x, g.y) = 1 \iff g.y \in \Gamma_{a,g.x} b$ which is if and only if there exists $\gamma \in \Gamma$ such that $\gamma a = g.x$ and $\gamma.b = g.y$. This is if and only if $g^{-1}\gamma \in \Gamma_{a,x}$ and $y = g^{-1}\gamma.b$, which is equivalent to $y \in \Gamma_{a,x} b$. So $f(g.x, g.y) = f(x,y)$.

Thus by assumption,

$$|\Gamma_{a,o} b| = \sum_x f(o,x) = \sum_x f(x,o) = \# \{x \ : \ o \in \Gamma_{a,x} b\}.$$

Since

$$\{x \ : \ o \in \Gamma_{a,x} b\} = \{x \ : \ \exists\, g \in \Gamma \ , \ g.a = x \ , \ g.b = o\} = \{g.a \ : \ \exists\, g \in \Gamma \ , \ g.b = o\} = \Gamma_{b,o} a,$$

we get that

$$|\Gamma_{a,o} b| = |\Gamma_{b,o} a|.$$

This was true for any a, b, o. Choosing $a = o$ we get that for any a, b, $|\Gamma_a b| = |\Gamma_{b,a} a| = |\Gamma_b a|$ by Proposition 11.1.5 ($\Gamma_{a,b} \neq \emptyset$ because of transitivity). This is the definition of unimodular. $\square$

We now turn to the proof of the general mass transport principle.

Proof of Theorem 11.1.1. If $y \in \Gamma_{x,b} a$ then $y = g.a$ for some $g \in \Gamma_{x,b}$. Since f is invariant this implies that $f(y,b) = f(g.a, g.x) = f(a,x)$. Thus,

$$\sum_{x \in \Gamma b} f(a,x) = \sum_{x \in \Gamma b} f(a,x) \sum_{y \in \Gamma_{x,b} a} \frac{1}{|\Gamma_{x,b} a|} = \sum_{x \in \Gamma b} \frac{1}{|\Gamma_{x,b} a|} \sum_{y \in \Gamma_{x,b} a} f(y,b)$$

We now interchange the sums. For this, note that

$$\begin{aligned}
\{(x,y) \ : \ x \in \Gamma b, y \in \Gamma_{x,b} a\} &= \big\{(x,y) \ : \ \exists\, g' \ , \ x = g'.b \ \text{ and } \ \exists\, g \ , \ g.x = b, g.a = y\big\} \\
&= \{(x,y) \ : \ \exists\, g \ , \ g.x = b, g.a = y\} \\
&= \big\{(x,y) \ : \ \exists\, g \ , \ g^{-1}.b = x, g.a = y\big\} \\
&= \{(x,y) \ : \ x \in \Gamma_{y,a} b, y \in \Gamma a\}
\end{aligned}$$

So

$$\sum_{x\in\Gamma b} f(a,x) = \sum_{x\in\Gamma b} \frac{1}{|\Gamma_{x,b}a|} \sum_{y\in\Gamma_{x,b}a} f(y,b) = \sum_{y\in\Gamma a} f(y,b) \sum_{x\in\Gamma_{y,a}b} \frac{1}{|\Gamma_{x,b}a|}$$

By Proposition 11.1.5, we know that for any $g \in \Gamma_{x,b}$, $\Gamma_{x,b} = \Gamma_b g$.

Let $x \in \Gamma_{y,a}b$. Then there exists g such that $g.y = a$ and $g.b = x$. So $g^{-1} \in \Gamma_{x,b}$ and $g^{-1}.a = y$. Thus, $\Gamma_{x,b} = \Gamma_b g^{-1}$ and so $\Gamma_{x,b}a = \Gamma_b g^{-1} a = \Gamma_b y$.

Plugging this into the above, we have

$$\begin{aligned}\sum_{x\in\Gamma b} f(a,x) &= \sum_{y\in\Gamma a} f(y,b) \sum_{x\in\Gamma_{y,a}b} \frac{1}{|\Gamma_{x,b}a|} \\ &= \sum_{y\in\Gamma a} f(y,b) \sum_{x\in\Gamma_{y,a}b} \frac{1}{|\Gamma_b y|} = \sum_{y\in\Gamma a} f(y,b) \cdot \frac{|\Gamma_{y,a}b|}{|\Gamma_b y|} = \sum_{y\in\Gamma a} f(y,b) \cdot \frac{|\Gamma_y b|}{|\Gamma_b y|},\end{aligned}$$

where the last equality is again by Proposition 11.1.5. $\square$

Finally, we may deduce that any group is a unimodular and transitive action on any one of its Cayley graphs.

> **Corollary 11.1.6** If G is a group generated by a finite symmetric set, then the action of G on the corresponding Cayley graph is transitive and unimodular.

Proof. Transitivity is simple.

As for unimodularity, we show that the mass transport principle is satisfied, and thus G is unimodular by Theorem 11.1.4.

Let $f : G \times G \to [0,\infty]$ be an invariant function. Then, since $x \mapsto gx^{-1}g$ is a bijection of G onto itself,

$$\sum_x f(g,x) = \sum_x f(gx^{-1}g, g) = \sum_{y=gx^{-1}g} f(y,g).$$

$\square$

Exercise 11.5 Let G be a finitely generated group.

Show that the action of G on the Cayley graph of G (by left multiplication, right Cayley graph) is unimodular.

Show that the full group of automorphisms of a Cayley graph acts by a unimodular action (*i.e.* the Cayley graph is unimodular). ◇◇◇

Exercise 11.6 Consider the following graph X. Start with the vertex set $V(X) = V(\mathbb{T})$ where $\mathbb{T} = \mathbb{T}_d$ is the d-regular tree ($d \geq 3$).

Fix an infinite simple path in the graph $\mathbb{T}$, say $p = (p_0, p_1, p_2, \ldots)$. We can define a *generation function*: For any vertex $x \in V(X)$ there is a unique closest point to x in the path p. Call this point $p(x)$. Note that $p(x) = x$ if and only if $x \in p$. Define $n(x)$ to be the unique integer such that $p(x) = p_{n(x)}$. Then let $||x|| = \mathrm{dist}(x, p) - n(x)$.

Now for any $x \in \mathbb{T}$ there is a unique vertex (x) such that $||(x)|| = ||x|| - 1$. This is the *parent* of x. Also, the vertex $((x))$ is the *grand-parent* of x, and is the unique vertex for which $||((x))|| = ||x|| - 2$.

For the edges of the graph X, take all edges in $\mathbb{T}$ (which connect every x to its parent), but also add an edge between every x and its grand-parent. That is,

$$E(X) = \big\{ \{x, ((x))\} \ , \ \{x, (x)\} \ : \ x \in \mathbb{T} \big\}.$$

This is known as the *grand-father graph*.

Prove the following:

- X is a transitive graph.
- The degree in X is $(d-1)^2 + d + 1 = d^2 - d + 2$.
- X is not unimodular.

◇◇◇

Exercise 11.7 Show that if you construct the grand-father graph from the 2-regular tree $\mathbb{T}_2$, you do obtain a Cayley graph. ◇◇◇

Exercise 11.8 Let X be a graph, and let $\Gamma \leq \mathrm{Aut}(X)$. Suppose that Γ acts transitively on X.

Show that of the action of Γ on X is unimodular, then X is a unimodular graph (*i.e.* the full automorphism group of X acts by a unimodular action). ◇◇◇

11.2 Applications of Mass Transport

11.2.1 Invariant Percolation

We now define a considerable generalization of bond / site percolation, but although it is very general, it is also extremely useful in many contexts.

Definition 11.2.1 (Invariant Percolation) Let G be a graph. Let $\mathbb{P}$ be any probability measure on either $2^{V(G)}, 2^{E(G)}$ or $2^{V(G)\cup E(G)}$. Let Ω be the canonical random subset. We say that $\mathbb{P}$ (or Ω) is an **invariant percolation** if $\mathbb{P}$ is invariant under the action of $\mathrm{Aut}(G)$; that is, if for every $\varphi \in \mathrm{Aut}(G)$, the probability measure $\mathbb{P} \circ \varphi^{-1} = \mathbb{P}$; in other words, $\varphi\Omega$ and Ω have the same distribution.

When we wish to distinguish between the cases $2^{V(G)}, 2^{E(G)}$ or $2^{V(G)\cup E(G)}$ above, we will say that $\mathbb{P}$ (or Ω) is an invariant site (resp. bond, resp. mixed) percolation.

Exercise 11.9 Let G be a Cayley graph. Show that both site and bond percolation on G are invariant percolation. ◇◇◇

Example 11.2.2 If G is transitive, there is no invariant percolation such that $|\Omega| = 1$ a.s.

Indeed, if the was such a measure $\mathbb{P}$, then $\mathbb{P}[\Omega = \{x\}]$ would be a constant independent of x (by transitivity), so

$$1 = \sum_x \mathbb{P}[\Omega = \{x\}] = \sum_x p = \infty!$$

△▽△

Example 11.2.3 Suppose G is transitive. Does there exists an invariant percolation Ω such that $0 < |\Omega| < \infty$ a.s.?

If there was such a measure $\mathbb{P}$, then define Ω' by first choosing Ω according to $\mathbb{P}$ and then choosing a uniformly chosen vertex x in Ω, and setting $\Omega' = \{x\}$. That is,

$$\mathbb{P}[\Omega' = \{x\} \mid \Omega] = \frac{\mathbf{1}_{\{x \in \Omega\}}}{|\Omega|}.$$

We claim that Ω' is an invariant percolation, which would contradict the previous example.

Indeed, if $\varphi \in \mathrm{Aut}(G)$ then

$$\mathbb{P}[\varphi\Omega' = \{x\} \,|\Omega] = \mathbb{P}[\Omega' = \left\{\varphi^{-1}x\right\} \mid \Omega] = \frac{\mathbf{1}_{\{\varphi^{-1}x \in \Omega\}}}{|\Omega|} = \frac{\mathbf{1}_{\{x \in \varphi\Omega\}}}{|\varphi\Omega|} = \mathbb{P}[\Omega' = \{x\} \mid \varphi\Omega].$$

Since Ω and $\varphi\Omega$ have the same distribution, this shows that Ω' is invariant. $\triangle \triangledown \triangle$

Example 11.2.4 If G is transitive then the number of finite components in an invariant percolation Ω must be 0 or ∞ a.s.

For suppose N is the number of finite components, and $\mathbb{P}[0 < N < \infty] > 0$. Then, if we set Ω' to be the union of all finite components, then the measure $\mathbb{P}[\Omega' \in \cdot | 0 < N < \infty]$ is an invariant percolation such that $\mathbb{P}[|\Omega'| < \infty | 0 < N < \infty] = 1$. $\triangle \triangledown \triangle$

Example 11.2.5 Let $\mathbb{P}$ be an invariant percolation, and let Ψ be the (random) set of all r-trifurcation points. Then, Ψ is an invariant percolation. So $|\Psi| \in \{0, \infty\}$ a.s. $\triangle \triangledown \triangle$

Proposition 11.2.6 Let $\mathbb{P}$ be an invariant percolation on some Cayley graph G. For any $x \in G$, the component of x, $\mathcal{C}(x)$, contains either 0 or infinitely many trifurcation points.

Proof. For any configuration $\omega \in \{0,1\}^{V(G)}$ and $x, y \in G$ define $F(x, y, \omega) = 0$ if the component of x in ω contains 0 or infinitely many r-trifurcation points. Otherwise, if the number of r-trifurcation points in the component of x in ω is N, set

$$F(x, y, \omega) = \begin{cases} \frac{1}{N} & \text{if } y \text{ is a trifurcation point} \\ 0 & \text{otherwise} \end{cases}$$

Note that $F(\varphi x, \varphi y, \varphi\omega) = F(x, y, \omega)$ for $\varphi \in \mathrm{Aut}(G)$. Thus, $f(x, y) := \mathbb{E}\, F(x, y, \Omega)$ is invariant. So we can apply Mass Transport to f and get

$$\sum_x f(o, x) = \sum_x f(x, o).$$

By definition, $\sum_x F(o, x, \omega) \in \{0, 1\}$ so $\sum_x f(o, x) \leq 1$.

However, for some configuration ω: Assume that $o \in G$ is such that o is a r-trifurcation point in ω, and the component of o in ω has finitely many r-trifurcation points. Then the component of o is infinite, and for any x in this component, $F(x, o, \omega) = \frac{1}{N}$ for some N.

We conclude that if $A = A_o$ is the event the the component of o contains finitely many r-trifurcation points, and that o is a r-trifurcation point, then for any $\omega \in A$, $\sum_x F(x, o, \omega) = \infty$. Thus, if $\mathbb{P}[A] > 0$ then

$$1 \geq \sum_x f(x, o) \geq \mathbb{E} \sum_x F(x, o, \omega) \mathbf{1}_{\{\omega \in A\}} = \infty.$$

Thus, by Mass Transport we must have that $\mathbb{P}[A_o] = 0$.

Now, let A'_o be the event that the component of o contains finitely many r-trifurcation points, then $\mathbb{P}[A'_o] \leq \sum_x \mathbb{P}[A_x] = 0$, because A'_o implies that there must be some x in the component of o that is a r-trifurcation point.

This holds for any o, so a.s. there are no components with finitely many r-trifurcation points. □

11.3 Critical Percolation on Non-Amenable Groups

Recall the definition of $\Phi(G)$ the Cheeger constant of a graph, and that G is amenable if and only if $\Phi(G) = 0$.

The following is a beautiful result of Benjamini, Lyons, Peres and Schramm.

Theorem 11.3.1 If G is a non-amenable unimodular transitive graph, then $\theta(p_c) = 0$.

The proof of this theorem is in two steps: first we show that the number of infinite clusters at p_c cannot be 1, and then we show that the number of infinite components at p_c cannot be ∞.

We work with bond percolation, the proofs for site percolation are similar.

Itai Benjamini

Russ Lyons

Yuval Peres

Oded Schramm (1961–2008)

11.3.1 No unique infinite component, $k_{p_c} \neq 1$

Lemma 11.3.2 If Ω is an invariant percolation on a transitive unimodular graph G such that all components of Ω are finite, then $\mathbb{E}[\deg_\Omega(x)] \leq \deg_G - \Phi(G)$.

Consequently, if $(\Omega_\varepsilon)_\varepsilon$ are a sequence of invariant bond percolation configurations such that $\lim_{\varepsilon \to 0} \mathbb{P}[\Omega_\varepsilon(e) = 1] = 1$, and if G is non-amenable, then there exists $\varepsilon > 0$ such that Ω_ε a.s. has an infinite component.

Proof. Note that for a finite set S,

$$\deg_G |S| - |\partial S| = \sum_{x \in S} \sum_{y \sim x} (1 - \mathbf{1}_{\{y \notin S\}}) = \sum_{x \in S} \deg_S(x).$$

Thus, for any finite set S,

$$\frac{1}{|S|} \sum_{x \in S} \deg_S(x) \leq \deg_G - \Phi(G).$$

Since all components of Ω are finite, define

$$F(x, y, \Omega) = \frac{\deg_\Omega(x)}{|\mathcal{C}(x)|} \cdot \mathbf{1}_{\{x \leftrightarrow y \text{ (in } \Omega) \}}.$$

So $f(x, y) := \mathbb{E}[F(x, y, \Omega)]$ is an invariant function. By the mass transport principle,

$$\mathbb{E}[\deg_\Omega(x)] = \sum_y f(x, y) = \sum_y f(y, x) = \mathbb{E}\, \tfrac{1}{|\mathcal{C}(x)|} \sum_{y \leftrightarrow x} \deg_\Omega(y) \leq \deg_G - \Phi(G).$$

Now if G is non-amenable, then the above estimate becomes non-trivial, and it is a uniform estimate over all invariant percolation configurations.

Thus, if we choose ε small enough so that $\mathbb{P}[\Omega_\varepsilon(e) = 1] > 1 - \deg_G^{-1} \Phi(G)$ then

$$\mathbb{E}[\deg_{\Omega_\varepsilon}(x)] = \sum_{y \sim x} \mathbb{P}[\Omega_\varepsilon(x \sim y) = 1] > \deg_G - \Phi(G),$$

which implies that Ω_ε cannot be composed of only finite components. $\square$

Theorem 11.3.3 Let G be a non-amenable unimodular transitive graph. Let k_{p_c} be the number of infinite components in bond percolation on G with parameter p_c. Then $\mathbb{P}[k_{p_c} = 1] = 0$.

Proof. Recall the natural coupling of percolation configurations Ω_p. If $k_{p_c} = 1$ then we can let Ω be the unique infinite component in Ω_{p_c}.

For a vertex x let $c(x)$ be the set of all vertices in Ω that are closest to x in the graph metric on G.

For $\varepsilon > 0$ define the following configuration $H_\varepsilon \in \{0,1\}^{E(G)}$: $H_\varepsilon(e) = 1$ if and only if $e = x \sim y$ such that

- $\mathrm{dist}_G(x, \Omega) < \varepsilon^{-1}, \mathrm{dist}_G(y, \Omega) < \varepsilon^{-1}$.
- For any $z \in c(x), w \in c(y)$, we have $\mathcal{C}_{p_c-\varepsilon}(z) = \mathcal{C}_{p_c-\varepsilon}(w)$.

Two observations: if $\varepsilon > \delta$ then $H_\varepsilon \leq H_\delta$. Also, for any $e = x \sim y$, there exists a small enough $\varepsilon > 0$ so that $H_\varepsilon(e) = 1$; this is because the sets $c(x), c(y)$ are finite, and in Ω, so for small enough ε they will be connected in $\Omega_{p_c-\varepsilon}$. **Here is where we use the assumption that** $k_{p_c} = 1$, in the fact that Ω is one big connected graph. Thus,

$$\lim_{\varepsilon \to 0} \mathbb{P}[H_\varepsilon(e) = 1] = 1.$$

By Lemma 11.3.2 this implies that for some small enough $\varepsilon > 0$, H_ε contains an infinite component, and consequently, an infinite simple path. Suppose that $(x_0, x_1, \ldots)$ is an infinite simple path such that $H_\varepsilon(x_j \sim x_{j+1}) = 1$ for all j.

First, for any j, $\mathrm{dist}(c(x_j), x_j) < \varepsilon^{-1}$, so $|c(x_j)| \leq |B(o, \varepsilon^{-1})|$, which is a uniform bound. Thus, there must be a subsequence $(j_k)_k$ such that $(c(x_{j_k}))_k$ are all distinct. That is $|\bigcup_j c(x_j)| = \infty$.

Also, for any j, we have that any two vertices $z \in c(x_j)$ and $w \in c(x_{j+1})$ are connected in $\Omega_{p_c-\varepsilon}$. So all $(c(x_j))_j$ must be in the same component of $\Omega_{p_c-\varepsilon}$. That is, $\bigcup_j c(x_j)$ is contained in a component of $\Omega_{p_c-\varepsilon}$.

So $\Omega_{p_c-\varepsilon}$ contains an infinite component, contradicting the definition of p_c. $\square$

11.3.2 No infinitely many infinite components, $k_{p_c} \neq \infty$

Recall the definition of a r-trifurcation point x; this event denoted by $\Psi_r(x)$. $\Psi_r(x) = \{M_r \geq 3\}$ where $M_r = (N_r)_{0,E(B(x,r))}$ and N_r is the number of infinite components intersecting the ball of radius r; that is, $\Psi_r(x)$ is the event that when forcing all edges in $B(x,r)$ to be closed, there are at least 3 infinite paths from $\partial B(x,r)$. This event is independent of $\mathcal{F}_{E(B(x,r))}$.

Lemma 11.3.4 Let Ω be an invariant percolation on a unimodular transitive graph G. Then, if $\mathbb{P}[\Psi_1(x), |\mathcal{C}_\Omega(x)| = \infty] > 0$ then

$$\mathbb{E}[\deg_\Omega(x) \mid |\mathcal{C}_\Omega(x)| = \infty] > 2.$$

Proof. Define $R(x,y,\Omega)$ to be the indicator of the event that there exists a *simple* infinite path of the form $(x = x_0, y = x_1, x_2, x_3, \ldots)$, such that $\Omega(x_n \sim x_{n+1}) = 1$ for all n. Set

$$F(x,y,\Omega) = \begin{cases} 2R(x,y,\Omega) & \text{if } R(y,x,\Omega) = 0, \\ R(x,y,\Omega) & \text{otherwise} \end{cases}$$

So F is invariant and thus also $f(x,y) = \mathbb{E}[F(x,y,\Omega)]$. (That is, $F(x,y,\Omega) = 1$ if one can go to infinity from x in two directions, one through y; if one can go to infinity through y but only in one direction, then $F(x,y,\Omega) = 2$.)

Note that $F(x,y,\Omega) + F(y,x,\Omega) = 2 \cdot \mathbf{1}_{\{\Omega(x\sim y)=1\}} \mathbf{1}_{\{|\mathcal{C}_\Omega(x)|=\infty\}}$, so

$$\sum_y F(x,y,\Omega) + F(y,x,\Omega) = 2\deg_\Omega(x)\mathbf{1}_{\{|\mathcal{C}_\Omega(x)|=\infty\}}.$$

The Mass Transport Principle now gives that

$$\mathbb{E}[\deg_\Omega(x)\mathbf{1}_{\{|\mathcal{C}_\Omega(x)|=\infty\}}] = \frac{1}{2}\sum_y f(x,y) + f(y,x) = \sum_y f(x,y) = \mathbb{E}\sum_y F(x,y,\Omega).$$

Whenever $|\mathcal{C}_\Omega(x)| = \infty$ and x is a trifurcation point, we must have that $\sum_y F(x,y,\Omega) \geq 3$. Also, for any x such that $|\mathcal{C}_\Omega(x)| = \infty$, there must exist at least on way of going from x to infinity, so $\sum_y F(x,y,\Omega) \geq 2$. Thus,

$$\begin{aligned} \mathbb{E}[\deg_\Omega(x)\mathbf{1}_{\{|\mathcal{C}_\Omega(x)|=\infty\}}] &= \sum_y \mathbb{E}[F(x,y,\Omega)\mathbf{1}_{\{\Psi_1(x),|\mathcal{C}_\Omega(x)|=\infty\}}] + \mathbb{E}[F(x,y,\Omega)\mathbf{1}_{\{\Psi_1(x)^c,|\mathcal{C}_\Omega(x)|=\infty\}}] \\ &\geq 3\,\mathbb{P}[\Psi_1(x), |\mathcal{C}_\Omega(x)| = \infty] + 2\,\mathbb{P}[\Psi_1(x)^c, |\mathcal{C}_\Omega(x)| = \infty] \\ &= 2\,\mathbb{P}[|\mathcal{C}_\Omega(x)| = \infty] + \mathbb{P}[\Psi_1(x), |\mathcal{C}_\Omega(x)| = \infty] > 2\,\mathbb{P}[|\mathcal{C}_\Omega(x)| = \infty]. \end{aligned}$$

$\square$

Lemma 11.3.5 Let Ω be an invariant percolation on a unimodular transitive graph G. Suppose that all components of Ω are trees (possibly finite) a.s. Then, if $\mathbb{P}[\Psi_1(x), x \leftrightarrow \infty] > 0$ then there a.s. exists some component $\mathcal{C}$ of Ω such that for Bernoulli bond percolation on $\mathcal{C}$, $p_c < 1$.

Proof. Since $\mathbb{P}[\Psi_1(x), x \leftrightarrow \infty] > 0$, we know that

$$m := \mathbb{E}[\deg_\Omega(x) \mid x \leftrightarrow \infty] > 2.$$

Choose $p < 1$ but large enough so that $mp > 2$ (it is important here that $m > 2$ and not just $m \geq 2$).

Let Ω' be Bernoulli bond p-percolation on Ω. Compute:

$$\begin{aligned}\mathbb{E}[\deg_{\Omega'}(x) \mid |\mathcal{C}_\Omega(x)| = \infty] &= \sum_{y \sim x} \mathbb{P}[\Omega(x \sim y) = 1 \mid |\mathcal{C}_\Omega(x)| = \infty]p \\ &= \mathbb{E}[\deg_\Omega(x) \mid |\mathcal{C}_\Omega(x)| = \infty]p = mp > 2.\end{aligned}$$

A combinatorial observation: If T is a finite tree then $|E(T)| = |T| - 1$. So

$$2|T| > 2|E(T)| = \sum_{x,y \in T} \mathbf{1}_{\{x \sim y\}} = \sum_{x \in T} \deg_T(x).$$

Now, if all components of Ω' are finite a.s., then we may define

$$F(x, y, \Omega) = \frac{\deg_{\Omega'}(x)}{|\mathcal{C}_{\Omega'}(x)|} \cdot \mathbf{1}_{\{y \in \mathcal{C}_{\Omega'}(x)\}} \mathbf{1}_{\{|\mathcal{C}_\Omega(x)| = \infty\}},$$

similarly to Lemma 11.3.2. As in that lemma, by mass transport we get that

$$\begin{aligned}\mathbb{E}[\deg_{\Omega'}(x) \mathbf{1}_{\{|\mathcal{C}_\Omega(x)| = \infty\}}] &= \mathbb{E}[\tfrac{1}{|\mathcal{C}_{\Omega'}(x)|} \sum_{y \in \mathcal{C}_{\Omega'}(x)} \deg_{\Omega'}(y) \cdot \mathbf{1}_{\{|\mathcal{C}_\Omega(x)| = \infty\}}] \\ &< 2\, \mathbb{P}[|\mathcal{C}_\Omega(x)| = \infty],\end{aligned}$$

since all components of Ω' are finite trees a.s.

Thus, because $\mathbb{E}[\deg_{\Omega'}(x) \mid |\mathcal{C}_\Omega(x)| = \infty] > 2$, we get that Ω' must contain some infinite component. Ω' is Bernoulli bond percolation on Ω, so some component of Ω must admit $p_c < 1$. $\square$

Definition 11.3.6 Recall the i.i.d. random variable $U(e)_e$ defining the natural coupling $(\Omega_p)_p$ of bond percolation. For p define $\mathsf{MSF} = \mathsf{MSF}_p$ to be the following configuration: $\mathsf{MSF}(x \sim y) = 1$ if and only if $\Omega_p(x \sim y) = 1$ and for any finite path $\gamma : x \to y$ such that $\gamma \subset \Omega_p$ there exists $0 \leq j \leq |\gamma| - 1$ such that $U(\gamma_j \sim \gamma_{j+1}) \geq U(x \sim y)$.

Exercise 11.10 Show that MSF is always a forest (*i.e.* does not contain a cycle). ◇◇◇

Exercise 11.11 Show that MSF is an invariant percolation. ◇◇◇

Lemma 11.3.7 If G is a transitive graph and $\mathsf{MSF} = \mathsf{MSF}_{p_c}$ for $p_c = p_c^{\mathsf{bond}}(G)$, then a.s. for every x the connected component $\mathcal{C}_{\mathsf{MSF}}(x)$ of x in MSF is a spanning tree of $\mathcal{C}_{p_c}(x)$.

Specifically, if x is a trifurcation point in an infinite component of Ω_{p_c} then it is also a trifurcation point in an infinite tree in MSF_{p_c}.

Proof. It suffices to prove that for any $x \sim y \in \Omega_{p_c}$. we have $y \leftrightarrow x$ in MSF.

Fix $x \sim y \in \Omega_{p_c}$. Let $u = U(x \sim y)$. We will show that if $x \not\leftrightarrow y$ in MSF then $\mathcal{C}_u(x)$ is infinite.

Assume that $y \not\leftrightarrow x$ in MSF.

Since $x \sim y \not\in \mathsf{MSF}$, it must be that there exists a path $\alpha : x \to y$ with $\alpha \subset \mathcal{C}_u(x)$.

Now, suppose we have some simple path $\alpha : x \to y$ with α open in Ω_u. If α is open in MSF then $x \leftrightarrow y$ in MSF, contradicting the assumption. Thus, there must exists an edge $\alpha_\ell \sim \alpha_{\ell+1} \not\in \mathsf{MSF}$. However, by definition this provides a simple path $\gamma : \alpha_\ell \to \alpha_{\ell+1}$ such that

$$\max_{0 \leq j \leq |\gamma|-1} U(\gamma_j \sim \gamma_{j+1}) < U(\alpha_\ell \sim \alpha_{\ell+1}) \leq u$$

(because α is open in Ω_u). So $E(\alpha \cup \gamma) \setminus \{\alpha_\ell \sim \alpha_{\ell+1}\}$ is open in Ω_u, and thus contains a simple path $\beta : x \to y$ which is open in Ω_u. In fact,

$$\beta = (\alpha_0, \ldots, \alpha_\ell = \gamma_0, \ldots, \gamma_{|\gamma|} = \alpha_{\ell+1}, \ldots, \alpha_{|\alpha|}),$$

which implies that $|\beta| = |\alpha| + |\gamma| - 1 > |\alpha|$.

By induction, this implies that for any k there is a path $\alpha : x \to y$ such that α is open in Ω_u and $|\alpha| > k$. So $|E(\mathcal{C}_u(x))| > k$ for any k, which implies that $|\mathcal{C}_u(x)| = \infty$.

We conclude if $x \sim y \in \Omega_{p_c}$ and $x \not\leftrightarrow y$ in MSF then $|\mathcal{C}_u(x)| = \infty$ for $u = U(x \sim y)$. Since $U(x \sim y) \neq p_c$ a.s., we have that

$$\mathbb{P}[x \sim y \in \Omega_{p_c} \text{ and } x \not\leftrightarrow y \text{ in } \mathsf{MSF}\ ] \leq \mathbb{P}[\exists\ u < p_c\ ,\ |\mathcal{C}_u(x)| = \infty] = 0.$$

So we have shown that a.s. for any $x \sim y \in \Omega_{p_c}$ also $x \leftrightarrow y$ in MSF. This implies by an exercise below that $\mathcal{C}_{\mathsf{MSF}}(x)$ is a spanning tree of $\mathcal{C}_{p_c}(x)$, for any x a.s.

Finally, if x is a trifurcation point in Ω_{p_c}, then $|\mathcal{C}_{p_c}(x)| = \infty$ and removing the edges adjacent to x would split $\mathcal{C}_{p_c}(x)$ into at least 3 infinite components. If $T(x)$ is the spanning tree of $\mathcal{C}_{p_c}(x)$ in MSF, then removing the edges adjacent to x would also split $T(x)$ into 3 infinite components. $\square$

Exercise 11.12 Show that if for any $x \sim y \in \Omega_p$ we have $x \leftrightarrow y$ in MSF_p, then for every connected component $\mathcal{C}_p(x)$ in Ω_p, the component $\mathcal{C}_{\mathsf{MSF}_p}(x)$ of x in MSF_p is a spanning tree of $\mathcal{C}_p(x)$. ◇◇◇

Exercise 11.13 Show that $\mathbb{P}[\exists\ u < p_c\ ,\ |\mathcal{C}_u(x)| = \infty] = 0$ ◇◇◇

Theorem 11.3.8 Let G be a non-amenable unimodular transitive graph. Let k_{p_c} be the number of infinite components in bond percolation on G with parameter p_c. Then, $\mathbb{P}[k_{p_c} = \infty] = 0$.

Proof. Recall the natural coupling of configurations Ω_p. We assume that $k_{p_c} = \infty$ for a contradiction.

Since $k_{p_c} = \infty$, we know that there exists $r > 0$ such that $\mathbb{P}_{p_c}[\Psi_r(x)] > 0$. Now the event $\Psi_r(x)$ is independent of $\mathcal{F}_{E(B(x,r))}$. Any configuration on the edges of $E(B(x,r))$ occurs with probability at least $q := (p_c \wedge (1 - p_c))^{|E(B(x,r))|} > 0$ (since $k_{p_c} = \infty$, we have $p_c < 1$). Thus, if we connect any three infinite components from $\partial B(x,r)$ by open paths inside $B(x,r)$, we get that $\mathbb{P}[\exists\ y \in B(x,r)\ :\ \Psi_1(y), y \leftrightarrow \infty \mid \Psi_r(x)] \geq q > 0$. This implies that there exists $y \in B(x,r)$ such that $\mathbb{P}[\Psi_1(y), y \leftrightarrow \infty] > 0$. Transitivity implies that this holds for any vertex, and specifically $\mathbb{P}[\Psi_1(x), x \leftrightarrow \infty] > 0$.

Now let $\mathsf{MSF} = \mathsf{MSF}_{p_c}$ as in Lemma 11.3.7. So with positive probability x is a trifurcation point in an infinite tree of MSF, by Lemma 11.3.7. Now, using Lemma 11.3.5, MSF a.s. contains some component $\mathcal{C}$ such that $p_c^{\mathsf{bond}}(\mathcal{C}) < 1$. But $\mathcal{C}$ is a subgraph of some connected component $\mathcal{C}'$ of Ω_{p_c}. So $p_c^{\mathsf{bond}}(\mathcal{C}') < 1$ as well. It is an exercise to show that this is impossible. $\square$

Exercise 11.14 Show that if $\mathcal{C}$ is a connected component of Ω_{p_c} then $p_c(\mathcal{C}) = 1$. ◇◇◇

Chapter 12

Site and Bond Percolation

12.1 $p_c^{\mathsf{bond}} \le p_c^{\mathsf{site}}$

Theorem 12.1.1 Let G be an infinite connect bounded degree graph. Then, for any $o \in G$, and $p \in (0,1)$, $\theta^{\mathsf{site}}_{G,o}(p) \le p\theta^{\mathsf{bond}}_{G,o}(p)$ and so $p_c^{\mathsf{bond}}(G) \le p_c^{\mathsf{site}}(G)$.

Proof. We couple a configuration $\omega \in \{0,1\}^{V(G)}$ with $\eta \in \{0,1\}^{E(G)}$ so that ω is site percolation, η is bond percolation, and if $o \leftrightarrow \infty$ in ω then also $\omega \leftrightarrow \infty$ in η.

Let $V(G) = \{o = x_0, x_1, \ldots, \}$ be some ordering of $V(G)$ and let $(B_e)_{e\in E(G)}$ be i.i.d. Bernoulli-p random variables.

Consider any configuration $\omega \in \{0,1\}^{V(G)}$. We define inductively a sequence of vertices $(v_k)_k$.

Set $V_1 = \{o\}, W_1 = \emptyset$, $n_1 = 0$ and $v_1 = x_{n_1} = o$.

Suppose $V_k \cup W_k = \{v_1, \ldots, v_k\}$ have been defined. Let n_{k+1} be the smallest index of a vertex $x \in \partial V_k \backslash (V_k \cup W_k)$. Set $v_{k+1} = x_{n_{k+1}}$. Let $e_{k+1} = v_j \sim v_{k+1}$ for $1 \le j \le k$ the smallest index such that $v_j \sim v_{k+1}$ and $v_j \in V_k$. Set $\eta(e_{k+1}) = \omega(v_{k+1})$. Set also

$$(V_{k+1}, W_{k+1}) = \begin{cases} (V_{k+1} \cup \{v_{k+1}\}, W_k) & \text{if } \omega(v_{k+1}) = 1, \\ (V_{k+1}, W_k \cup \{v_{k+1}\}) & \text{if } \omega(v_{k+1}) = 0. \end{cases}$$

If there does not exist n_{k+1}, *i.e.* if $\partial V_k \setminus (V_k \cup W_k) = \emptyset$ then stop the procedure.

Once the procedure terminates, for all $e \in E(G)$ such that $\eta(e)$ is not defined let $\eta(e) = B_e$.

Note that if ω is p-site percolation then η is p-bond percolation.

Also, if $|\mathcal{C}_\omega(o)| = \infty$, then at each step we may find a new vertex $x \in \partial V_k \setminus (V_k \cup W_k)$ that has $\omega(x) = 1$, so for some $j > k$ we will have $\omega(v_j) = 1$ and $v_j \in V_j$. That is, $|\mathcal{C}_\omega(o)| = \infty$ implies that $|V_k| \to \infty$. Since $V_k \subset \mathcal{C}_\eta(o)$ for all k, we get that $|\mathcal{C}_\omega(o)| = \infty$ implies that $|\mathcal{C}_\eta(o)| = \infty$.

Moreover, if $|\mathcal{C}_\omega(o)| = \infty$ then $\omega(o) = 1$. Since η is independent of $\omega(o)$, we have that

$$\theta^{\mathsf{site}}_{G,o}(p) = \mathbb{P}[|\mathcal{C}_\omega(o)| = \infty] \le \mathbb{P}[\omega(o) = 1, |\mathcal{C}_\eta(o)| = \infty] = p\theta^{\mathsf{bond}}_{G,o}(p).$$

□

12.2 Bounds for Site in Terms of Bond

Lemma 12.2.1 Let G be an infinite connected graph of bounded degree. Suppose that $(\Omega(x))_{x\in V(G)}$ are independent Bernoulli random variables such that $\sup_x \mathbb{E}[\Omega(x)] = p$. Then, there exists a coupling of Ω and Ω_p such that $\Omega \le \Omega_p$ (where Ω_p is p-site percolation).

Proof. For every x let $p_x = \mathbb{E}[\Omega(x)]$. Let $(U(x))_x$ be independent $U[0,1]$ random variables. Set $\Omega'(x) = \mathbf{1}_{\{U(x)\le p_x\}}$ and $\Omega_p(x) = \mathbf{1}_{\{U(x)\le p\}}$.

It is immediate that $\Omega' \le \Omega_p$.

Also, Ω' has the same law as Ω. □

Theorem 12.2.2 Let G be an infinite connect graph of maximal degree d. Then for any $p \in (0,1)$ and $o \in G$,

$$\theta^{\mathsf{bond}}_{G,o}(p) \le \theta^{\mathsf{site}}_{G,o}(1 - q^{d-1})$$

where $q = 1 - p$. Consequently, $p_c^{\mathsf{site}} \leq 1 - \left(1 - p_c^{\mathsf{bond}}\right)^{d-1}$.

Proof. Let $V(G) = \{o = x_0, x_1, x_2, \ldots\}$ be some ordering of $V(G)$. Let $\eta \in \{0,1\}^{E(G)}$. Let $(B_x)_{x \in V(G)}$ be i.i.d. Bernoulli-$(1 - q^{d-1})$ random variables.

Start with $V_0 = W_0 = \emptyset$. If $\eta(x_1, y) = 0$ for all $y \sim x_1$ then stop the procedure with the empty sequence, and let $\omega(x) = B_x$ for all x. If there exists n_1 such that $\eta(x_1 \sim x_{n_1}) = 1$ then set $v_1 = x_{n_1}$ for n_1 the smallest such index, and $V_1 = \{o, v_1\}, W_1 = \emptyset$.

Given $V_k \cup W_k = \{o, v_1, \ldots, v_k\}$ choose n_{k+1} to be the smallest index such that $x_{n_{k+1}} \in \partial V_k \setminus (V_k \cup W_k)$. If no such vertex exists, then stop the procedure. When such n_{k+1} exists, set $v_{k+1} = x_{n_{k+1}}$.

If there exists $y \notin V_k \cup W_k$ such that $\eta(v_{k+1} \sim y) = 1$ then set $\omega(v_{k+1}) = 1$. Otherwise set $\omega(v_{k+1}) = 0$. Also set

$$(V_{k+1}, W_{k+1}) = \begin{cases} (V_{k+1} \cup \{v_{k+1}\}, W_k) & \text{if } \omega(v_{k+1}) = 1, \\ (V_{k+1}, W_k \cup \{v_{k+1}\}) & \text{if } \omega(v_{k+1}) = 0. \end{cases}$$

The important observation here is the following: Conditioned on V_k, W_k, v_{k+1}, the conditional probability that $\omega(v_{k+1}) = 1$ is $1 - q^{d'} \leq 1 - q^{d-1}$ for some $d' \leq d - 1$.

Finally, for all x such that $\omega(x)$ is not defined, set $\omega(x) = B_x$.

Now, if η is p-bond percolation then $(\omega(x))_x$ are independent Bernoulli random variables, with $\mathbb{E}[\omega(x)] \leq 1 - q^{d-1}$ for all x.

Also, if $|\mathcal{C}_\eta(o)| = \infty$, then at each step we can always find an edge $x \sim y$ such that $x \in \partial V_k \setminus (V_k \cup W_k)$, $y \notin V_k \cup W_k$, and $\eta(x \sim y) = 1$. So at some $j > k$ we will have $\omega(v_j) = 1$ and $v_j \in V_j$. Since $V_k \subset \mathcal{C}_\omega(o)$ for all k, we have that $|\mathcal{C}_\eta(o)| = \infty$ implies that $|\mathcal{C}_\omega(o)| = \infty$.

Thus,

$$\theta_{G,o}^{\mathsf{bond}}(p) = \mathbb{P}[|\mathcal{C}_\eta(o)| = \infty] \leq \mathbb{P}[|\mathcal{C}_\omega(o)| = \infty] \leq \theta_{G,o}^{\mathsf{site}}(1 - q^{d-1}),$$

the last inequality by Lemma 12.2.1.

We conclude that if $p' := 1 - (1-p)^{d-1} > 1 - (1 - p_c^{\mathsf{bond}})^{d-1}$ then $p > p_c^{\mathsf{bond}}$, so $\theta_{G,o}^{\mathsf{site}}(p') > 0$ and $p_c^{\mathsf{site}} \leq p' = 1 - (1-p)^{d-1}$. Taking infimum over all such p' completes the proof. $\square$

Let us summarize these two sections with:

Theorem 12.2.3 Let G be an infinite connected graph of maximal degree d. Then,

$$p_c^{\mathsf{bond}} \leq p_c^{\mathsf{site}} \leq 1 - (1 - p_c^{\mathsf{bond}})^{d-1}.$$

Chapter 13

Uniqueness and Amenability

13.1 Uniqueness Phase

Recall our coupling of all percolation spaces together by assigning i.i.d. uniform-$[0,1]$ random variables $(U(x))_x$ to the vertices of G, and the configurations $\Omega_p(x) = \mathbf{1}_{\{U(x) \leq p\}}$, so that Ω_p has law $\mathbb{P}_p^{\mathsf{site}}$. Let us also use the notation $\mathcal{C}_p(x)$ to denote the component of x in the subgraph Ω_p.

For a configuration ω we define $J(\omega)$ to be the union of all infinite components in ω.

Proposition 13.1.1 If G is a transitive graph and $q \geq p > 0$ such that $\theta(p) > 0$, then $D = \mathrm{dist}(\mathcal{C}_q(x), J(\Omega_p))$ satisfies

$$\mathbb{P}[D \leq 2 \mid |\mathcal{C}_q(x)| = \infty] = 1.$$

Proof. We will define a process which will be helpful in analyzing the infinite components of percolation. This process is called **invasion percolation**. We will define it for site percolation, the corresponding process for bond percolation being an obvious analogue.

Consider $(U_x)_{x \in G}$ i.i.d. random variables each uniform on $[0,1]$. Define a growing sequence of subsets $(I_n(x))_n$ as follows. Start with $I_0(x) = \{x\}$. Given $I_n(x)$ define $I_{n+1}(x) = I_n(x) \cup \{y\}$ where $y \in \partial I_n(x)$ is the vertex adjacent to $I_n(x)$ with the minimal label U_y. That is, $U_y < U_z$ for all $z \in \partial I_n(x)$. (Recall $\partial S = \{y \sim S \ : \ y \notin S\}$.) Set $I(x) = \bigcup_n I_n(x)$.

Now, fix $\varepsilon > 0$ and let $r = r(\varepsilon) > 0$ be large enough so that $\mathbb{P}_p[B_r \leftrightarrow \infty] \geq 1 - \varepsilon$.

For a finite set of vertices $S \subset G$ let $A(S)$ be the event that $\partial S \leftrightarrow \infty$ in Ω_p. Note that $A(S) \in \mathcal{T}_S$. By transitivity, if $B_r(y) \subset \bar{S}$ then $\mathbb{P}[A(\bar{S})] \geq 1 - \varepsilon$ (where $\bar{S} = S \cup \partial S$). Also, $\{I_k(x) = S\} \in \mathcal{F}_{\bar{S}}$.

Let

$$\Gamma_r = \{S \subset G \mid |S| < \infty \ , \ \exists \, y \in G \ , \ B_r(y) \subset S\}.$$

Thus, for any $S \in \Gamma_r$ we have

$$\mathbb{P}[A(\bar{I}_k(x)) \ , \ I_k(x) = S] = \mathbb{P}[A(\bar{S})] \cdot \mathbb{P}[I_k(x) = S] \geq (1-\varepsilon) \cdot \mathbb{P}[I_k(x) = S],$$

and summing over all $S \in \Gamma_r$ we get

$$\mathbb{P}[A(\bar{I}_k(x))] \geq \mathbb{P}[A(\bar{I}_k(x)) \ , \ I_k(x) \in \Gamma_r] \geq (1-\varepsilon) \cdot \mathbb{P}[I_k(x) \in \Gamma_r].$$

Also, by an exercise below, there exist k such that $\mathbb{P}[I_k(x) \in \Gamma_r] \geq 1 - \varepsilon$. Thus,

$$\mathbb{P}[A(\bar{I}_k(x))] \geq (1-\varepsilon)^2.$$

Finally, note that $A(\bar{I}_k(x))$ implies that $\mathrm{dist}(I(x), J(\Omega_p)) \leq \mathrm{dist}(I_k(x), J(\Omega_p)) \leq 2$. Thus,

$$\mathbb{P}[\mathrm{dist}(I(x), J(\Omega_p)) \leq 2] \geq (1-\varepsilon)^2 \to 1,$$

as the left-hand side does not depend on $\varepsilon > 0$.

Since $I(x) \subset \mathcal{C}_q(x)$ whenever $\mathcal{C}_q(x)$ is infinite (exercise), the proof is complete because

$$\mathbb{P}[\mathrm{dist}(\mathcal{C}_q(x), J(\Omega_p)) \leq 2 \bigm| |\mathcal{C}_q(x)| = \infty] \\ \geq \mathbb{P}[\mathrm{dist}(I(x), J(\Omega_p)) \leq 2 \bigm| |\mathcal{C}_q(x)| = \infty] = 1.$$

□

Exercise 13.1 Show that if $\mathcal{C}_p(x)$ is infinite then $I(x) \subset \mathcal{C}_p(x)$. ◇◇◇

Solution to ex:1. :(
If $I(x) \not\subset \mathcal{C}_p(x)$, then there exists a minimal k such that $y \in I_k(x) \setminus I_{k-1}(x)$ and $y \notin \mathcal{C}_p(x)$. Note that since k was a minimal such time, $I_{k-1}(x) \subset \mathcal{C}_p(x)$. Thus, if $U_y \leq p$ then $y \in \mathcal{C}_p(x)$ as well, since $y \sim I_{k-1}(x)$. So it must be that $U_y > p$.

If $\mathcal{C}_p(x) \not\subset I_{k-1}(x)$, then there exists $z \in \partial I_{k-1}(x)$ that is connected to x, *i.e.* $z \in \mathcal{C}_p(x)$. Specifically, $U_z \leq p < U_y$, contradicting the fact that $y \in I_k(x)$.

We conclude that it must be that for this minimal k we have $I_{k-1}(x) = \mathcal{C}_p(x)$, implying that $\mathcal{C}_p(x)$ is finite.
:) ✓

Exercise 13.2 Show that for any $\varepsilon > 0$ and any radius $r > 0$ there exist k such that $\mathbb{P}[\exists\ y \in G\ ,\ B_r(y) \subset I_k(x)] \geq 1 - \varepsilon$. ◇◇◇

Solution to ex:2. :(
Fix $\varepsilon > 0$ and $r > 0$. Let $S_m = \{y\ :\ \mathrm{dist}(x,y) = m\}$. For ever n let

$$\tau_n = \inf\{k\ :\ \mathrm{dist}(I_k(x), S_{2n(r+1)}) = r + 1\}.$$

Note that since $|I_k(x)| = k + 1$, it must be that $\tau_n < |B_{2n(r+1)}|$. Choose some $Y_n = Y_n(I_{\tau_n}(x))$ so that $\mathrm{dist}(I_{\tau_n}(x), Y_n) = r+1$ and $Y_n \in S_{2n(r+1)}$. Note that $B_r(Y_n) \cap \partial I_{\tau_n}(x) \neq \emptyset$. Also, note that $I_{\tau_n}(x), \tau_n, Y_n$ are all measurable with respect to $\mathcal{F}_{B_{(2n-1)(r+1)}(x)}$.

Let

$$A_n = \{\forall\ z \in B_r(Y_n)\ ,\ U_z \leq \tfrac{p_c}{2}\}.$$

Since $\{Y_n = y\}, A_{n-1}, \ldots, A_1 \in \mathcal{F}_{B_{(2n-1)(r+1)}}$ (which is independent of $\mathcal{F}_{B_r(y)}$ for any $y \in S_{2n(r+1)}$),

$$\begin{aligned}&\mathbb{P}[A_n \mid Y_n = y\ ,\ A_{n-1}^c, \ldots, A_1^c] \\ &\qquad = \mathbb{P}[\forall\ z \in B_r(y)\ ,\ U_z \leq \tfrac{p_c}{2} \mid Y_n = y\ ,\ A_{n-1}^c, \ldots, A_1^c] = \alpha := \left(\tfrac{p_c}{2}\right)^{|B_r|} > 0.\end{aligned}$$

Averaging over $y \in S_{2n(r+1)}$, we have that $\mathbb{P}[\bigcap_n A_n^c] = 0$.

Now, since $B_r(Y_n) \cap \partial I_{\tau_n}(x) \neq \emptyset$, the event A_n implies that for all $k > \tau_n$, either $B_r(Y_n) \subset I_k(x)$ or there exists $z \in B_r(Y_n) \cap \partial I_k(x)$. Thus, if $B_r(Y_n) \not\subset I(x)$ then it must be that for all $k > 0$ we added a vertex $v_k \in I_{\tau_n+k}(x) \setminus I_{\tau_n+k-1}(x)$ and there exists some $v_k \neq z \in B_r(Y_n) \cap \partial I_{\tau_n+k-1}(x)$. But this implies that $U_{v_k} < U_z \leq \frac{p_c}{2}$. Note that the set $I(x) \setminus I_{\tau_n}(x) = \{v_1, v_2, \ldots\}$ is an infinite set, which is composed of an infinite connected set $I(x)$ with a finite set $I_{\tau_n}(x)$ removed. This implies that $I(x) \setminus I_{\tau_n}(x) = \{v_1, v_2, \ldots\}$ contains some infinite component, $\mathcal{C} \subset \{v_1, v_2, \ldots\}$ whose labels are all $U_v \leq \frac{p_c}{2}, v \in \mathcal{C}$. This has probability 0 since $\frac{p_c}{2} < p_c$.

So if we let $\mathcal{E}$ be the probability 1 event that all components in $\Omega_{p_c/2}$ are finite, we have that $\mathcal{E} \cap A_n \subset \{B_r(y) \subset I(x)\}$. However, this gives

$$\mathbb{P}[\exists\ y\ ,\ B_r(y) \subset I(x)] \geq \mathbb{P}[\bigcup_n A_n \cap \mathcal{E}] = 1.$$

:) ✓

Exercise 13.3 Let $q > p$ be such that $\theta(p) > 0$. Recall that $J(\Omega_p)$ is the union of all infinite components of Ω_p.

Show that if $\mathrm{dist}(\mathcal{C}_q(x), J(\Omega_p)) \leq 1$ then $\mathcal{C}_q(x)$ contains an infinite component of Ω_p. ◇◇◇

A technical lemma we will require, which is a property sometimes known as *tolerance*.

We work in some product space $[0,1]^V$ (*e.g.* when $V = V(G)$). The cylinder σ-algebras $\mathcal{F}_S$ are defined as in the $\{0,1\}^V$ case, except with configurations in $[0,1]^V$. For any subset S define

$$A_S := \left\{ \omega \in [0,1]^V \ : \ \exists \ \omega' \in A \ , \ \omega\big|_S = \omega'\big|_S \right\} = \bigcup_{\omega \in A} C_{\omega,S}.$$

So $A_S \in \mathcal{F}_S$. Since $A = A_S \cap A_{V \setminus S}$, and since $\mathcal{F}_S, \mathcal{F}_{V\setminus S}$ are independent, we get that $\mathbb{P}[A] = \mathbb{P}[A_S] \cdot \mathbb{P}[A_{V\setminus S}]$. So if $\mathbb{P}[A] > 0$ then both $\mathbb{P}[A_S] > 0$ and $\mathbb{P}[A_{V\setminus S}] > 0$.

Lemma 13.1.2 **(Tolerance Lemma)** Let S be a finite set and let $0 \le p < q \le 1$. Let A be some event and let $A_{S,p,q} = A_{V\setminus S} \cap \{\forall \ s \in S \ , \ U(s) \in (p,q]\}$.

If $\mathbb{P}[A] > 0$ then $\mathbb{P}[A_{S,p,q}] > 0$.

That is, the vertex labels in some finite set can be changed to be in $[p,q]$ without ruining the positivity of the given event.

Proof. Since S is finite, and since $\{\forall \ s \in S \ , \ U(s) \in [p,q]\} \in \mathcal{F}_S$,

$$\mathbb{P}[A_{V\setminus S}, \forall \ s \in S \ U(s) \in [p,q]] = \mathbb{P}[A_{V\setminus S}] \cdot (q-p)^{|S|}$$

which is positive whenever $\mathbb{P}[A] > 0$ and $q > p$ and $|S| < \infty$. □

Theorem 13.1.3 Let G be a transitive graph.

Under the standard coupling of percolation, for any $q > p$ such that $\theta(p) > 0$, any infinite component of Ω_q must contain an infinite component of Ω_p.

Consequently, if Ω_p has a unique infinite component a.s. then so does Ω_q.

Proof. Fix x. Let $D = \mathrm{dist}(\mathcal{C}_q(x), J(\Omega_p))$. By an exercise above, it suffices to prove that

$$\mathbb{P}[D \le 1 \ \big| \ |\mathcal{C}_q(x)| = \infty] = 1.$$

By the invasion percolation argument we know that $\mathbb{P}[D \leq 2 \mid |\mathcal{C}_q(x)| = \infty] = 1$.

Let $J = J(\Omega_p)$ and let $M = \mathcal{C}_q(x) \setminus \bar{J}$ (where $\bar{J} = J \cup \partial J$ and $\partial J = \{y \notin J \ : \ y \sim J\}$). Consider the set $Y = \partial J \cap \partial M$. Every $y \in Y$ admits $y \sim \mathcal{C}_q(x)$ and $y \sim J$. Moreover, since $Y \subset \partial J$, we have that conditioned on $(U_z)_{z \notin \bar{J}}$ and Ω_p, the distribution of $(U_y)_{y \in Y}$ is that of independent random variables, each uniform on $(p, 1]$. (Indeed, they are condition to be in ∂J, so cannot be at most p.)

Now, assume there exists $y \in Y$ such that $U_y \leq q$. Because $y \in \partial \mathcal{C}_q(x)$, this implies that $y \in \mathcal{C}_q(x)$. Because $y \in \partial J$, we can conclude that $D \leq 1$. That is,

$$\{D > 1\} \subset \bigcap_{y \in Y} \{U_y > q\}.$$

We may thus compute,

$$\mathbb{P}[D > 1 \bigm| \Omega_p \ , \ U_z, z \notin \bar{J}] \leq \left(\tfrac{1-q}{1-p}\right)^{|Y|} \leq \mathbf{1}_{\{|Y| < \infty\}},$$

and by taking expectations,

$$\begin{aligned} \mathbb{P}[D > 1 \ , \ |\mathcal{C}_q(x)| = \infty] &= \mathbb{P}[D > 1 \ , \ |M| = \infty] \\ &\leq \mathbb{P}[|Y| < \infty \ , \ |M| = \infty] \leq \mathbb{P}[|Y| < \infty \ , \ |\mathcal{C}_q(x)| = \infty]. \end{aligned}$$

(We have used that on the event $D > 1$ we have $M = \mathcal{C}_q(x)$.)

Finally, we show that $\mathbb{P}[|Y| < \infty \ , \ |\mathcal{C}_q(x)| = \infty] = 0$.

Indeed, assume for a contradiction that $\mathbb{P}[|Y| = n \ , \ |\mathcal{C}_q(x)| = \infty] > 0$ for some $n \geq 1$. This implies that there exist $y_1, \ldots, y_n \in G$ such that for the event $A = \{Y = \{y_1, \ldots, y_n\} \ , \ |\mathcal{C}_q(x)| = \infty\}$ we have $\mathbb{P}[A] > 0$. Let $Z = \overline{\{y_1, \ldots, y_n\}} = \{y_j \ , \ z \sim y_j \ : \ j = 1, \ldots, n\}$. However, by the Tolerance Lemma, we may force $U_z > p$ for all $z \in Z$ and still keep a positive probability, so

$$0 < \mathbb{P}[A_{Z,p,1}] \leq \mathbb{P}[D > 2 \ , \ |\mathcal{C}_q(x)| = \infty],$$

which contradicts the argument given by invasion percolation.

Also, $|Y| = 0$ implies $D > 2$, so again $\mathbb{P}[|Y| = 0 \ , \ |\mathcal{C}_q(x)| = \infty] = 0$.

We conclude that

$$\mathbb{P}[|Y| < \infty \ , \ |\mathcal{C}_q(x)| = \infty] = 0.$$

$\square$

Recall the number k_p of infinite components which is a.s. constant, and a.s. in the set $\{0, 1, \infty\}$. Theorem 13.1.3 tells us that if $p < q$ and $k_p = 1$ then $k_q = 1$ as well. So the quantity

$$p_u = p_u(G) := \inf\{p \ : \ k_p = 1\} = \sup\{p \ : \ k_p = \infty\}$$

is well defined. Thus, there exists at most three phases: no infinite components, then infinitely many infinite components, then a unique infinite component.

There is a lot of research when one of these phases is empty. It is also very difficult to understand what happens at the critical points.

Of course, we have already seen that $p_u = p_c$ when the graph is amenable. In the non-amenable case, it may be that $p_u = 1$ so we always have infinitely many infinite components.

Exercise 13.4 Show that for the d-regular tree $p_u(\mathbb{T}_d) = 1$. ◇◇◇

13.2 Uniqueness and Amenability

Recall that the Burton-Keane Theorem tells us that $p_c = p_u$ for any amenable transitive graph. Complementing the amenable case, is the following theorem by Pak and Smirnova-Nagnibeda:

Theorem 13.2.1 Let G be a non-amenable finitely generated group. Then there exists a Cayley multi-graph such that on this multi-graph $p_c^{\mathsf{bond}} < p_u^{\mathsf{bond}}$.

✓ A caveat, perhaps, is that the Cayley graph constructed may be a multi-graph. This is the reason we work with bond percolation, since site percolation does not care about multiple edges.

It is in fact a conjecture by Benjamini and Schramm that any non-amenable Cayley graph has $p_c < p_u$. That is, together with the Burton-Keane Theorem, the conjecture becomes: G is amenable if and only if $p_c = p_u$.

13.3 Percolation and Expansion

Recall the definition $\partial S = \{x \notin S \ : \ x \sim S\}$ and $\Phi = \Phi(G) = \inf \frac{|\partial S|}{|S|}$ where the infimum is over finite connected subsets of G. A non-amenable graph is such that $\Phi > 0$. In other words, any subset S (perhaps not connected) admits $|\partial S| \geq \Phi |S|$.

Theorem 13.3.1 For percolation on G,

$$p_c(G) \leq \frac{1}{1+\Phi(G)}.$$

✓ So any non-amenable graph has a non-trivial estimate $p_c < 1$.

Proof. Enumerate the vertices of G by $V(G) = \{o = v_1, v_2, \ldots, \}$. Let ω be a configuration sampled from $\mathbb{P}_p$; so $(\omega(v_k))_k$ are i.i.d. Bernoulli-p random variables.

Define a sequence of random variables $(a_k)_k$ as follows. Set $a_1 = \omega(o) = \omega(v_1)$ and $n_1 = 1$. Given that we have defined $a_1, \ldots, a_k$ for some $k \geq 1$, define $a_{k+1} = v_{n_{k+1}}$ where n_{k+1} is chosen as follows: We always denote $V_k = \{n_1, \ldots, n_k\}$; these are the indices of the vertices that have already been examined. Also denote $O_k = \{v_m \ : \ m \in V_k, \omega(v_m) = 1\}$; these are the open vertices that have already been examined.

Case 1: $\partial O_k \not\subset \{v_m \ : \ m \in V_k\}$. Then there exists a vertex v_m such that $m \notin V_k$ and $v_m \sim O_k$. Let n_{k+1} be the smallest index such that $v_{n_{k+1}} \sim O_k$ and $n_{k+1} \notin V_k$.

Case 2: $\partial O_k \subset \{v_m \ : \ m \in V_k\}$. Then, set n_{k+1} to be the smallest index not in V_k.

This defines a sequence of i.i.d. Bernoulli-p random variables $(a_k)_k$.

Suppose that for any $m \in V_k$ we have that $|\mathcal{C}(v_m)| < \infty$. Then at some time $j > k$ we must have examined all vertices in $\mathcal{C}(v_m)$, so there is a $j > k$ such that $\partial O_j \subset \{v_i \ : \ i \in V_j\}$.

Thus, if there are no infinite components, then there is an infinite sequence of times $(k_j)_j$ such that for all j, $\partial O_{k_j} \subset \left\{ v_i \ : \ i \in V_{k_j} \right\}$.

Now, if $\partial O_k \subset \{ v_m \ : \ m \in V_k \}$ then $O_k \uplus \partial O_k \subset \{ v_m \ : \ m \in V_k \}$, so

$$(1+\Phi)|O_k| \leq |O_k| + |\partial O_k| \leq |V_k| = k.$$

Also, $\sum_{j=1}^k a_j = |O_k|$. So if $\partial O_k \subset \{ v_m \ : \ m \in V_k \}$ then

$$\frac{1}{k} \sum_{j=1}^k a_j \leq \frac{|O_k|}{(1+\Phi)|O_k|} = \frac{1}{1+\Phi}.$$

We conclude that the event that all components are finite implies that there exists an infinite sequence $(k_j)_j$ such that for all j,

$$\frac{1}{k_j} \sum_{i=1}^{k_j} a_i \leq \frac{1}{1+\Phi}.$$

That is, $\{ \forall \ x \ , \ x \not\leftrightarrow \infty \} \subset \left\{ \liminf \frac{1}{k} \sum_{j=1}^k a_j \leq \frac{1}{1+\Phi} \right\}$. Now, the law of large numbers states that $\frac{1}{k} \sum_{j=1}^k a_j$ converges a.s. to p. Thus, for any $p > \frac{1}{1+\Phi}$ we get that

$$\mathbb{P}[\forall \ x \ , \ x \not\leftrightarrow \infty] \leq \mathbb{P}[\liminf \tfrac{1}{k} \textstyle\sum a_j \leq \tfrac{1}{1+\Phi}] \leq \mathbb{P}[\liminf \tfrac{1}{k} \textstyle\sum a_j < p] = 0.$$

That is, $p_c \leq \frac{1}{1+\Phi}$.

Now, there was nothing special here about site percolation. For bond percolation we could have defined a_k by exploring the edges coming out of O_k. $\square$

13.4 Uniqueness and Self-Avoiding Polygons

A **self avoiding polygon** is a simple cycle in the graph. That is, a path $\gamma = (\gamma_0, \ldots, \gamma_n)$ such that $\gamma_i \neq \gamma_j$ for all $0 \leq i \neq j < n$ and $\gamma_0 = \gamma_n$.

Fix some vertex $o \in G$ and let SAP_n be the set of all self avoiding polygons γ of length n, with $\gamma_0 = o$. Let SAW_n be the set of all self-avoiding paths of length n started at o. (Recall that a self avoiding path γ is a path such that $\gamma_i \neq \gamma_j$ for all $i \neq j$.)

We will be interested in convergence of the series $\sum_n |\mathsf{SAW}_n| z^n$. The Cauchy-Hadamard theorem gives us the rate of convergence of this series: it is the reciprocal of

$$\mu(G) := \limsup_{n\to\infty} |\mathsf{SAW}_n|^{\frac{1}{n}},$$

which we called the *connective constant*.

Exercise 13.5 Show that $|\mathsf{SAP}_n| \leq |\mathsf{SAW}_{n-1}|$. ◇◇◇

Lemma 13.4.1 Let $p < \frac{1}{\mu(G)}$. Then,

$$\lim_{r\to\infty} \sup_{x\notin B(o,r)} \mathbb{P}_p[o \leftrightarrow x] = 0.$$

Consequently, it is impossible that $p > p_u$ for such a p, and so $p_u \geq \frac{1}{\mu(G)}$.

Proof. Let $CV_p(o,x)$ be the set of open pivotal vertices for $o \leftrightarrow x$ (in the percolation Ω_p). That is, $v \in CV_p(o,x)$ if

$$\{o \leftrightarrow x\} \cap \{o \not\leftrightarrow x\}_{0,v} = \{v \text{ is open }\} \cap \{o \leftrightarrow x\}_{1,v} \cap \{o \not\leftrightarrow x\}_{0,v}.$$

Define

$$m_k(r) = \mathbb{P}_p[\exists x \notin B(o,r)\ ,\ |CV_p(o,x)| \leq k, o \leftrightarrow x].$$

Note that if $o \leftrightarrow x$ then $o,x \in CV_p(o,x)$. So if $|CV_p(o,x)| = 2$ then $CV_p(o,x) = \{o,x\}$ and there must exist two disjoint paths $\gamma,\gamma' : o \to x$ that are open in Ω_p; otherwise there would be another pivotal vertex for o,x. (This is Menger's theorem.) That is, o,x are in a Ω_p-open self avoiding polygon, which is just the composition of γ and the reverse of γ'. So we conclude that $m_k(r) = 0$ for $k < 2$, and

$$\begin{aligned} m_2(r) &\leq \mathbb{P}[o \text{ is in a } \Omega_p\text{-open self avoiding polygon of length at least } 2r] \\ &\leq \sum_{n\geq 2r} |\mathsf{SAP}_n| p^{n+1} \leq \sum_{n\geq 2r} |\mathsf{SAW}_{n-1}| p^{n+1}. \end{aligned}$$

When $p < \frac{1}{\mu(G)}$ we get that the above series is summable and so $m_2(r) \to 0$ as $r \to \infty$.

Let $R > r$ and $x \notin B(o,R)$. Suppose that $o \leftrightarrow x$. If $|CV_p(o,x)| \leq k+1$, then

- Case 1: $CV_p(o,x) \cap B(o,r) = \{o\}$. So there exists $y \notin B(o,r)$ such that $CV_p(o,y) = \{o,y\}$ and $y \leftrightarrow o$.
- Case 2: $|CV_p(o,x) \cap B(o,r)| \geq 2$. Then $|CV_p(o,x) \cap B(o,r)^c| \leq k-1$. So there must exist $o \neq y \in B(o,r)$ such that $|CV_p(y,x)| \leq k$ and $y \leftrightarrow x$.

We conclude that

$$m_{k+1}(R) \leq m_2(r) + |B(o,r)| \cdot m_k(R-r).$$

A simple induction on k now shows that $m_k(r) \to 0$ as $r \to \infty$. Indeed, this holds for $k=2$. For general k: for any $\varepsilon > 0$ there exists r so that $m_2(r) < \frac{\varepsilon}{2}$ and also there exists $R > r$ large enough so that $m_k(R-r) < |B(o,r)|^{-1}\frac{\varepsilon}{2}$, so that $m_k(R) < \varepsilon$. This completes the induction.

We conclude that for any $r > 0$ and $k \geq 2$, for any $x \notin B(o,r)$,

$$\mathbb{P}_p[x \in \mathcal{C}_q(o)] \leq m_k(r) + \mathbb{P}[x \in \mathcal{C}_q(o), |CV_p(o,x)| \geq k+1].$$

Note that for $q < p$,

$$\mathbb{P}[\forall\ v \in C\ ,\ U(v) \leq q \mid \Omega_p] = \mathbf{1}_{\{\forall\ v \in C\ ,\ \Omega_p(v)=1\}} \cdot \left(\tfrac{q}{p}\right)^{|C|}.$$

So averaging over $\Omega_p \in \{CV_p(o,x) = C\}$,

$$\mathbb{P}[\forall\ v \in C\ ,\ U(v) \leq q \mid CV_p(o,x) = C] \leq \left(\tfrac{q}{p}\right)^{|C|}.$$

Summing over $|C| \geq k+1$,

$$\mathbb{P}[x \in \mathcal{C}_q(o), |CV_p(o,x)| \geq k+1] \leq \left(\tfrac{q}{p}\right)^{k+1}.$$

For any $\varepsilon > 0$ take k large enough so that $\left(\frac{q}{p}\right)^{k+1} < \frac{\varepsilon}{2}$ and then r large enough so that $m_k(r) < \frac{\varepsilon}{2}$. So for any $\varepsilon > 0$ there exists $r > 0$ large enough so that

$$\sup_{x \notin B(o,r)} \mathbb{P}[x \in \mathcal{C}_q(o)] < \varepsilon.$$

This holds for all $q < p < \frac{1}{\mu}$. Which proves the first assertion.

For the second assertion, let $p > p_u$. Then by Harris' Lemma, because there is a unique infinite component,

$$\mathbb{P}_p[o \leftrightarrow x] \geq \mathbb{P}_p[o \leftrightarrow \infty, x \leftrightarrow \infty] \geq \mathbb{P}_p[o \leftrightarrow \infty] \cdot \mathbb{P}_p[x \leftrightarrow \infty] \geq \theta(p)^2 > 0.$$

If $p < \frac{1}{\mu}$ then taking x farther and farther away from o would make the left-hand side go to 0, which is impossible. So it must be that $p \geq \frac{1}{\mu}$. Thus, $p_u \geq \frac{1}{\mu}$.

This completes the proof for site percolation.

For bond percolation the proof is similar, and is given as an exercise. $\square$

Exercise 13.6 Consider bond percolation on a transitive graph G. Let $CE_p(x,y)$ be the set of open pivotal edges for $x \leftrightarrow y$: that is, $e \in CE_p(x,y)$ if e is open and $x \leftrightarrow y$ and closing e gives a configuration where $x \not\leftrightarrow y$. Let

$$m_k(r) = \mathbb{P}_p[\exists\ x \notin B(o,r)\ ,\ |CE_p(o,x)| \leq k\ ,\ x \leftrightarrow o].$$

Show that if $p < \frac{1}{\mu}$ then $m_k(r) \to 0$ as $r \to \infty$, for any fixed k.

Prove that $p_u \geq \frac{1}{\mu}$. ◇◇◇

13.5 Random Walks

A simple random walk on a graph G is a sequence $(X_t)_t$ of vertices such that $\mathbb{P}[X_{t+1} = y \mid X_t = x] = \mathbf{1}_{\{y \sim x\}} \frac{1}{\deg_G(x)}$. This is a whole area of research, but let us mention just those properties we require.

Given a random walk on G, one may define

$$\rho(G) := \limsup_{t \to \infty} (\mathbb{P}[X_t = o \mid X_0 = o])^{\frac{1}{t}}.$$

This is easily shown to exist by Fekete's Lemma. It is also simple to show that

$$\rho(G)^k = \limsup_{t \to \infty} (\mathbb{P}[X_{kt} = o \mid X_0 = o])^{\frac{1}{t}}.$$

It is immediate that $0 \leq \rho(G) \leq 1$.

Kesten's Thesis includes the following theorem:

Theorem 13.5.1 Let G be a transitive d-regular graph. Then G is non-amenable if and only if $\rho(G) < 1$. In fact,

$$\frac{\Phi(G)^2}{2d^2} \leq 1 - \sqrt{1 - \tfrac{\Phi(G)^2}{d^2}} \leq 1 - \rho(G) \leq \frac{\Phi(G)}{d}.$$

We may thus use Kesten's Thesis to connect amenability and random walks. For our purposes, note that if $(X_t)_t$ is a random walk on G, then

$$\mathbb{P}[X_t = o \mid X_0 = o] \geq \sum_{\gamma \in \mathsf{SAP}_t} \mathbb{P}[(X_0, \ldots, X_t) = \gamma] = |\mathsf{SAP}_t| \cdot (\deg_G)^{-t}.$$

Thus, $\nu(G) \leq \deg_G \cdot \rho(G)$.

Now, if $\rho(G) \leq \frac{1}{2}$ then by Kesten's Thesis, $\Phi(G) \geq \frac{\deg_G}{2} \geq \deg_G \cdot \rho(G) \geq \nu(G)$. So

$$p_c(G) \leq \frac{1}{1 + \Phi(G)} < \frac{1}{\nu(G)} \leq p_u(G).$$

So to prove the Pak-Smirnova-Nagnibeda Theorem it suffices to show that G admits a Cayley graph with $\rho(G) \leq \frac{1}{2}$.

If we take $G^{(k)}$ to be the graph G with an edge $x \sim y$ for any path $\gamma : x \to y$ such that $|\gamma| = k$. Then the random walk on $G^{(k)}$ has the same distribution as $(X_{kt})_t$, so $\rho(G^{(k)}) = \rho(G)^k$. If k is large enough so that $\rho(G)^k \leq \frac{1}{2}$ (which is possible whenever G is non-amenable and $\rho < 1$) then in $G^{(k)}$ we have $p_c^{\mathsf{bond}} < p_u$.

It is also quite simple to see that if $G = \langle S \rangle$ for finite $S = S^{-1}$ then $G^{(k)}$ for odd k is a Cayley graph of G with respect to the multi-set $\{s_1 s_2 \cdots s_k \ : \ s_j \in S\}$, and that this multi-set generates G when k is odd.

Chapter 14

Percolation on Finite Graphs

14.1 Galton-Watson Processes Conditioned on Extinction

Let X be a random variable with values in $\mathbb{N}$. We assume that $\mathbb{P}[X = 0] + \mathbb{P}[X = 1] < 1$. Recall that the Galton-Watson Process with offspring distribution X, GW_X, is the process $(Z_n)_n$ such that $Z_0 = 0$ and $Z_{n+1}|Z_n$ is the sum of Z_n independent copies of X.

We have already seen that if $q = q(X)$ is the extinction probability for GW_X, then q is the smallest solution in $(0, 1]$ to the equation $q = \mathbb{E}[q^X]$, and $q = 1$ if and only if $\mathbb{E}[X] \leq 1$.

For such a random variable X, let us define X^* to be the random variable with values in $\mathbb{N}$ and density given by

$$\mathbb{P}[X^* = x] = \mathbb{P}[X = x]q^{x-1}.$$

One may check that indeed

$$\sum_x \mathbb{P}[X^* = x] = \mathbb{E}[q^{X-1}] = 1.$$

Lemma 14.1.1 Let $(Z_n)_n$ be GW_X for some X with $\mathbb{E}[X] > 1$. Let $q = q(X)$ be the extinction probability. Let E be the event of extinction (*i.e.* $E = \{\exists\, n : \; Z_n = 0\}$ and $q = \mathbb{P}[E]$). Then, conditioned on E, the process $(Z_n)_n$ has the same distribution as GW_{X^*}.

Proof. Let $(Z_n)_n$ be GW_X and let $(Z_n^*)_n$ be GW_{X^*}. We have to show that for every sequence $(1 = z_0, z_1, \ldots, z_t)$,

$$\mathbb{P}[Z[0,t] = z[0,t] \mid E] = \mathbb{P}[Z^*[0,t]], \tag{14.1}$$

where $Z[s,t] = (Z_s, \ldots, Z_t)$, $Z^*[s,t] = (Z_s^*, \ldots, Z_t^*)$ and $z[s,t] = (z_s, \ldots, z_t)$.

Using Bayes and the Markovian property of the Galton-Watson process,

$$\mathbb{P}[Z[0,t] = z[0,t] \mid E] = \frac{\mathbb{P}[E \mid Z[0,t] = z[0,t]] \cdot \mathbb{P}[Z[0,t] = z[0,t]]}{\mathbb{P}[E]} = \mathbb{P}[Z[0,t] = z[0,t]]q^{z_t - 1}.$$

We now prove (14.1) by induction on t. For $t = 0$ this is obvious, since $Z_0 = Z_0^* = 1$.

For $t > 0$, for any $(z_0, \ldots, z_{t-1}, z_t)$, using $x = z_{t-1}, y = z_t$, by induction,

$$\begin{aligned}
\mathbb{P}[Z[0,t] = z[0,t] \mid E] &= \mathbb{P}[Z[0,t] = z[0,t]]q^{y-1} \\
&= q^{y-1} \cdot \mathbb{P}[Z[0,t-1] = z[0,t-1]] \cdot \mathbb{P}[Z_t = y \mid Z_{t-1} = x] \\
&= q^{y-1} \cdot q^{1-x}\, \mathbb{P}[Z[0,t-1] = z[0,t-1] \mid E] \cdot \mathbb{P}[Z_t = y \mid Z_{t-1} = x] \\
&= q^{y-x} \cdot \mathbb{P}[Z^*[0,t-1] = z[0,t-1]] \cdot \mathbb{P}[Z_t = y \mid Z_{t-1} = x].
\end{aligned}$$

Since

$$\mathbb{P}[Z^*[0,t] = z[0,t]] = \mathbb{P}[Z^*[0,t-1] = z[0,t-1]] \cdot \mathbb{P}[Z_t^* = y \mid Z_{t-1}^* = x],$$

it suffices to prove that for any x, y and any $t > 0$,

$$q^{y-x} \cdot \mathbb{P}[Z_t = y \mid Z_{t-1} = x] = \mathbb{P}[Z_t^* = y \mid Z_{t-1}^* = x]. \tag{14.2}$$

On $Z_{t-1} = x$, Z_t is the sum of x independent copies of X. On $Z_{t-1}^* = x$, Z_t^* is the sum of x independent copies of X^*. Let $(X_j)_j$ be i.i.d. copies of X and $(X_j^*)_j$ i.i.d. copies of X^*.

We prove (14.2) by induction on x. For $x = 0$ it is clear that (14.2) holds.

For $x = 1$,

$$q^{y-x}\, \mathbb{P}[Z_t = y \mid Z_{t-1} = x] = q^{y-1}\, \mathbb{P}[X_1 = y] = \mathbb{P}[X^* = y] = \mathbb{P}[X_1^* = y].$$

For $x > 1$, let $Y = \sum_{j=2}^{x} X_j$ and $Y^* = \sum_{j=2}^{x} X_j^*$. By induction, for any $k \le y$,

$$\mathbb{P}[Y = k] = \mathbb{P}[Z_t = k \mid Z_{t-1} = x-1] = q^{x-1-k}\,\mathbb{P}[Z_t^* = k \mid Z_{t-1}^* = x-1] = q^{x-1-k}\,\mathbb{P}[Y^* = k],$$

so

$$\begin{aligned}
\mathbb{P}[Z_t = y \mid Z_{t-1} = x] = \mathbb{P}[X_1 + Y = y] &= \sum_{k=0}^{y} \mathbb{P}[X_1 = y-k] \cdot \mathbb{P}[Y = k] \\
&= \sum_{k=0}^{y} q^{1+k-y}\,\mathbb{P}[X_1^* = y-k] \cdot q^{x-1-k}\,\mathbb{P}[Y^* = k] \\
&= q^{x-y}\,\mathbb{P}[X_1^* + Y^* = y] = q^{x-y}\,\mathbb{P}[Z_t^* = y \mid Z_{t-1}^* = x].
\end{aligned}$$

□

Example 14.1.2 Recall that if $f(s) = \mathbb{E}[s^X]$ then f is convex, so has an increasing derivative on $(0,1)$. Also, for X^* as above,

$$\mathbb{E}[s^{X^*}] = \sum_x \mathbb{P}[X = x] q^{x-1} s^x = q^{-1} f(sq).$$

So

$$\mathbb{E}[X^*] = \frac{\partial}{\partial s} q^{-1} f(sq)\Big|_{s=1} = f'(q).$$

Note that since f' is strictly increasing on $(0,1)$, it must be that $f'(q) < f'(s)$ for all $s > q$. The function $f(s) - s$ obtains minimum on $[q,1]$ at some $s \in (q,1)$ such that $f'(s) = 1$ (because $f(q) - q = 0 = f(1) - 1$ and f is strictly convex on $(q,1)$). Thus, $f'(q) < f'(s) = 1$. △▽△

Example 14.1.3 Let us consider the Poisson case.

If $X \sim \mathrm{Poi}(\lambda)$ then

$$\mathbb{E}[s^X] = \sum_{k=0}^{\infty} e^{-\lambda} \frac{s^k \lambda^k}{k!} = e^{-\lambda(1-s)}.$$

So $q = q(X)$ is the solution to $e^{-\lambda(1-q)} = q$ or equivalently, $qe^{-\lambda q} = e^{-\lambda}$.

Note that the function $x \mapsto xe^{-x}$ has a maximum at $x = 1$ and increases in $(0,1)$ and decreases in $(1,\infty)$. Thus, for every $\lambda > 1$ we may define λ^* as the unique $\lambda^* < 1$ such that $\lambda^* e^{-\lambda^*} = \lambda e^{-\lambda}$.

If $\mathbb{E}[X] > 1$ and $X \sim \mathrm{Poi}(\lambda)$ then

$$\mathbb{P}[X^* = x] = e^{-\lambda}\frac{\lambda^x}{x!} \cdot q^{x-1} = e^{-\lambda q}\frac{(\lambda q)^x}{x!},$$

so $X^* \sim \mathrm{Poi}(q\lambda)$. Note that $q\lambda e^{-q\lambda} = \lambda e^{-\lambda}$ so $q\lambda = \lambda^*$ and $X^* \sim \mathrm{Poi}(\lambda^*)$.
△▽△

Example 14.1.4 How about the Binomial case?

If $X \sim \mathrm{Bin}(n, p)$, then

$$\mathbb{E}[s^X] = \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} s^k = (1-p+sp)^n = (1-p(1-s))^n.$$

So $q = (1-p(1-q))^n$.

As for X^*,

$$\mathbb{P}[X^* = x] = \binom{n}{k} p^k (1-p)^{n-k} q^{k-1} = \binom{n}{k}\left(\tfrac{pq}{1-p(1-q)}\right)^k \cdot \left(\tfrac{1-p}{1-p(1-q)}\right)^{n-k},$$

so $X^* \sim \mathrm{Bin}(n, \frac{pq}{1-p(1-q)})$.

Note that if $p = \frac{1+\varepsilon}{n}$ then $(1-p(1-s))^n$ is very close to $e^{-(1+\varepsilon)(1-s)}$ when n is very large. So q is close to the solution of $e^{-(1+\varepsilon)(1-s)} = s$, as in the Poisson-$(1+\varepsilon)$ case.

This case is interesting, because as we have seen, p-bond percolation on the rooted d-regular tree is a Galton-Watson process with offspring distribution $X \sim \mathrm{Bin}(d-1, p)$, in the sense that the component of the root is exactly those vertices in such a process.

We used this to deduce that $p_c = \frac{1}{d-1}$ and that $\theta(p_c) = 0$. We will also use this point of view to calculate the distribution of the component size.

△▽△

14.2 Exploration of a Galton-Watson Tree

Suppose we consider a Galton-Watson tree with offspring distribution X. Here is another way to sample the tree, but with a time change.

Instead of generating the tree one generation every time step, we will generate it by letting every active particle reproduce at different time steps.

Let $X_1, X_2, \ldots,$ be independent copies of X. Start with $Y_0 = 1$. Here Y_t denotes the number of "active" particles (that is those that have not yet reproduced).

At every time step $t > 0$, if $Y_{t-1} > 0$ let $Y_t = Y_{t-1} + X_t - 1$. That is, one particle reproduces with X_t offspring and dies. If $Y_{t-1} = 0$ then set $Y_t = 0$.

Note that as long as there is no extinction (*i.e.* $Y_t > 0$) the number of dead particles is always t. If extinction occurs, then there is a time t for which there are no active particles, *i.e.* all particles are dead. If $T = \inf\{t : Y_t = 0\}$ then the total number of offspring is T.

Now, let $\mathbb{T}$ be the rooted d-regular tree rooted at o. If we were exploring the component of o in p-bond percolation on $\mathbb{T}$, then the size of this component is the total number of offspring in a Galton-Watson process with offspring distribution $X \sim \mathrm{Bin}(d-1, p)$, which has the distribution of T above. This point of view gives us a simple way of computing the size distribution of the component of o.

Proposition 14.2.1 For $X \sim \mathrm{Bin}(d-1, p)$ and $(Y_t)_t$ as above we have that if $Y_{t-1} > 0$ then $Y_t \sim \mathrm{Bin}(t(d-1), p) - (t-1)$ and $Y_t \in \sigma(X_1, \ldots, X_t)$.

Proof. By induction on t. For $t = 1$ this is clear from $Y_1 = X_1$.

For the induction step: if $Y_t > 0$ then $Y_{t+1} = Y_t + X_{t+1} - 1$. By induction, since it must be that $Y_{t-1} > 0$ (otherwise $Y_t = 0$) we have $Y_t \sim \mathrm{Bin}(t(d-1), p) - (t-1)$ and is measurable with respect to $\sigma(X_1, \ldots, X_t)$. Since X_{t+1} is independent of $\sigma(X_1, \ldots, X_t)$, and since the sum of independent binomials is also binomial, we have $Y_{t+1} \sim \mathrm{Bin}(t(d-1) + (d-1), p) - t$, which is the proposition. $\square$

Another result we require is a large deviations result for binomial random variables.

Proposition 14.2.2 Let $B \sim \mathrm{Bin}(n, p)$ then for any $0 < \varepsilon < 1$,

$$\max\{\mathbb{P}[B \geq (1+\varepsilon)np], \mathbb{P}[B \leq (1-\varepsilon)np]\} \leq \exp\left(-\tfrac{1}{4}\varepsilon^2 np\right).$$

Proof. We start by calculating the exponential moment of B.

$$\mathbb{E}[e^{\alpha B}] = \sum_{k=0}^{n} \binom{n}{k} p^k (1-p)^{n-k} e^{\alpha k} = (1 - p(1-e^{\alpha}))^n.$$

Thus, for $0 < \alpha \leq \frac{1}{2}$, by Markov's inequality,

$$\begin{aligned}\mathbb{P}[B \geq (1+\varepsilon)np] = \mathbb{P}[e^{\alpha B} \geq e^{\alpha(1+\varepsilon)np}] &\leq (1 + p(e^{\alpha} - 1))^n e^{-\alpha(1+\varepsilon)np} \\ &\leq \exp\left(np(\alpha + \alpha^2 - \alpha(1+\varepsilon))\right),\end{aligned}$$

where we have used the inequalities $1 + \xi \leq e^{\xi}$ and $e^{\xi} \leq 1 + \xi + \xi^2$ which is valid for all $\xi \leq \frac{1}{2}$. Optimizing over α we choose $\alpha = \frac{\varepsilon}{2}$ to obtain $\alpha^2 - \varepsilon\alpha = -\frac{\varepsilon^2}{4}$ and

$$\mathbb{P}[B > (1+\varepsilon)np] \leq \exp\left(-\tfrac{1}{4}\varepsilon^2 np\right).$$

For the other direction we use negative exponents, so that for $\alpha > 0$

$$\begin{aligned}\mathbb{P}[B \leq (1-\varepsilon)np] = \mathbb{P}[e^{-\alpha B} \geq e^{-\alpha(1-\varepsilon)np}] &\leq (1 - p(1 - e^{-\alpha}))^n e^{\alpha(1-\varepsilon)np} \\ &\leq \exp\left(-np(\alpha - \alpha^2 - \alpha(1-\varepsilon))\right),\end{aligned}$$

using the same inequalities as before (here $-\alpha < 0$ so there is no restriction). A similar optimization gives $\alpha = \frac{\varepsilon}{2}$, so $\varepsilon\alpha - \alpha^2 = \frac{\varepsilon^2}{4}$ and

$$\mathbb{P}[B \leq (1-\varepsilon)np] \leq \exp\left(-\tfrac{1}{4}\varepsilon^2 np\right).$$

□

Proposition 14.2.3 For p-bond percolation on $\mathbb{T}$ and $\mathcal{C} = \mathcal{C}(o)$:

- If $p < \frac{1}{d-1}$ then
$$\mathbb{P}_p[|\mathcal{C}| > t] \leq e^{-\alpha t}$$
for $\alpha = \frac{(1-(d-1)p)^2}{(d-1)p}$.
- If $p > \frac{1}{d-1}$ then $\mathbb{P}_p[|\mathcal{C}| = \infty] \geq 1 - q$, where $q = q(\mathrm{Bin}(d-1, p))$ and
$$\mathbb{P}_p[t < |\mathcal{C}| < \infty] \leq e^{-\beta t}$$
where $\beta = \frac{(1-\mu)^2}{\mu}$ and $\mu = \mathbb{E}[X^*] = (d-1)\frac{pq}{1-p(1-q)}$.

Proof. We begin with the sub-critical case, $p < \frac{1}{d-1}$.

We consider the Galton-Watson process $(Y_t)_t$ described above with offspring distribution $X \sim \text{Bin}(d-1,p)$, and $T = \inf\{t : Y_t = 0\}$ so that $|\mathcal{C}|$ and T have the same distribution.

Now,

$$\mathbb{P}[T > t] \leq \mathbb{P}[Y_t > 0] = \mathbb{P}[\text{Bin}(t(d-1),p) > t].$$

Since $p < \frac{1}{d-1}$, taking $\varepsilon = \frac{t-t(d-1)p}{t(d-1)p} = \frac{1-(d-1)p}{(d-1)p} > 0$ we obtain by large deviations of binomials that

$$\mathbb{P}[T > t] = \mathbb{P}[\text{Bin}(t(d-1),p) > t(d-1)p(1+\varepsilon)] \leq \exp\left(-\varepsilon^2 t(d-1)p\right) = \exp\left(-\frac{(1-(d-1)p)^2}{(d-1)p} \cdot t\right).$$

If $p > \frac{1}{d-1}$ then the Galton-Watson process is super critical and q is just the extinction probability, so $\mathbb{P}[T = \infty] \geq 1 - q$. Also, if E is the event of extinction,

$$\mathbb{P}[t < T < \infty] = \mathbb{P}[t < T < \infty \mid E] \cdot q \leq \mathbb{P}[t < T < \infty \mid E].$$

Now conditioned on E we have a Galton-Watson process with offspring distribution $X^* \sim \text{Bin}(d-1,p^*)$, where $p^* = \frac{pq}{1-p(1-q)} = pq^{1-\frac{1}{d-1}}$. The important thing here is that the expectation is $\mu = (d-1)p^* < 1$. So we can use the previous result on sub-critical processes so that

$$\mathbb{P}[t < T < \infty] \leq \exp\left(-\frac{(1-\mu)^2}{\mu} \cdot t\right).$$

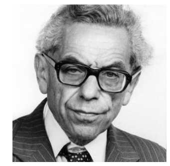
□

14.3 The Erdös-Rényi Random Graph

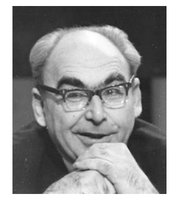
Definition 14.3.1 For a positive integer n and parameter $p \in [0,1]$ we define $G(n,p)$ to be the random graph on vertex set $V(G(n,p)) = \{1,2,\ldots,n\}$ whose edge set is given by letting $x \sim y$ with probability p independently for all $x \neq y \in V(G(n,p))$.

That is, $G(n,p)$ is just the random subgraph of the complete graph on

$\{1,\ldots,n\}$, obtained by p-bond percolation.



Paul Erdös (1913–1996)



Alfréd Rényi (1921–1970)

This model was introduced by Erdös and Rényi in 1960. It is the analogue of bond percolation in the finite setting. There are many interesting phenomena regarding the Erdös-Rényi model, let us elaborate on one:

For the random graph $G(n,p)$ let $\mathcal{C}_1 \geq \mathcal{C}_2 \geq \ldots \geq \mathcal{C}_N$ denote the sizes of the connected components in $G(n,p)$ in non-increasing order (so N is the number of connected components in $G(n,p)$).

Theorem 14.3.2 For any $\varepsilon > 0$ there exist constants $c, C > 0$ such that the following holds.

- If $p(n) = \frac{1-\varepsilon}{n}$ then as $n \to \infty$,
$$\mathbb{P}_{n,p(n)}[\mathcal{C}_1 \leq C \log n] \to 1.$$
- If $p(n) = \frac{1+\varepsilon}{n}$ then as $n \to \infty$,
$$\mathbb{P}_{n,p(n)}[\mathcal{C}_1 \geq cn \ , \ \forall\, j \geq 2 \ , \ \mathcal{C}_j \leq C \log n] \to 1.$$

That is, with high probability, in the *super-critical phase*, $p > \frac{1}{n}$, the largest component is of linear size, but all other components are at most logarithmic in size. In the *sub-critical phase*, $p < \frac{1}{n}$, all components are at most logarithmic in size.

14.4 Sub-Critical Erdös-Rényi

Let us define an exploration of the component of 1 in $G(n,p)$. In this process, all vertices are either live (L_t), dead (D_t) or neutral (N_t).

We start with $L_0 = \{1\}, D_0 = \emptyset, N_0 = \{2,\ldots,n\}$. At step $t > 0$, we choose some live vertex, say $x \in L_{t-1}$. We add x to the dead vertices, $D_t = D_{t-1} \cup \{x\}$. We then consider all $y \in N_{t-1}$ and let $y \in L_t$ if $x \sim y$ is open, and $y \in N_t$ if $x \sim y$ is closed. That is, let

$$L_t = L_{t-1} \setminus \{x\} \bigcup \{y \in N_{t-1} \ : \ x \sim y \in G(n,p)\}$$

and

$$N_t = \{y \in N_{t-1} \ : \ x \sim y \notin G(n,p)\} .$$

If $L_t = \emptyset$ the process terminates.

Now, if $Y_t := |L_t|$ then since no edge is examined more than once, we have that $Y_t = Y_{t-1} - 1 + \mathrm{Bin}(N_{t-1}, p)$. Since $|D_t| = t$ and $|N_t| = n - |D_t| - |L_t| = n - t - Y_t$ by construction, we have that $Y_t = Y_{t-1} + Z_t - 1$ where $Z_t \sim \mathrm{Bin}(n - t + 1 - Y_{t-1}, p)$ are all independent.

Note that the size of the component of 1 is $T := \inf\{t \ : \ Y_t = 0\}$. Moreover, given $(L_t, D_t, N_t)_{t \le T}$ all the edges $x \sim y$ with $x, y \notin \mathcal{C}(1)$ are independent, so the remaining subgraph on $\{1, 2, \ldots, n\} \setminus \mathcal{C}(1)$ is independently distributed as $G(n - |\mathcal{C}(1)|, p)$.

Lemma 14.4.1 There exists a coupling of $(L_t, D_t, N_t)_t$ above with a Galton-Watson process $(\tilde{Y}_t)_t$ of offspring distribution $\mathrm{Bin}(n,p)$ such that $Y_t = |L_t| \le \tilde{Y}_t$ for all t.

Specifically, for any $\varepsilon > 0$, if $p(n) \le \frac{1-\varepsilon}{n}$ then as $n \to \infty$,

$$\mathbb{P}_{n,p(n)}[\mathcal{C}_1 > \tfrac{1}{\varepsilon^2} \log n] \to 0.$$

Proof. If we write $N_t = \{y_1 < y_2 < \cdots < y_k\}$ for $k = n - t - Y_t$, we can define

$$X_{t+1} := Y_{t+1} - Y_t + 1 + \mathrm{Bin}(n-k, p) = \sum_{j=1}^{k} \mathbf{1}_{\{x \sim y_j \text{ is open }\}} + \mathrm{Bin}(n-k, p) \sim \mathrm{Bin}(n,p),$$

conditional on $(L_s, D_s, N_s)_{s \le t}$. So inductively, if $(\tilde{Y}_t)_t$ is the Galton-Watson process $\tilde{Y}_t = \tilde{Y}_{t-1} + X_t - 1$, since $\tilde{Y}_t - \tilde{Y}_{t-1} = X_t - 1 \ge Y_t - Y_{t-1}$, we obtain that $Y_t \le \tilde{Y}_t$ for all t.

Moreover, if $T > t$ then $Y_t > 0$ and so $\tilde{Y}_t > 0$ and $\tilde{T} > t$. Hence $T \le \tilde{T}$.

We get that if $np < 1$ then

$$\mathbb{P}[\tilde{T} > k] \le e^{-\alpha k},$$

where $\alpha = \frac{(1-np)^2}{np}$. If $p(n) \le \frac{1-\varepsilon}{n}$ then $\alpha \ge \gamma := \frac{\varepsilon^2}{1-\varepsilon}$. Since all vertices in $G(n,p)$ have the same distribution

$$\mathbb{P}_{n,p(n)}[\mathcal{C}_1 > k] \le \sum_x \mathbb{P}_{n,p(n)}[|\mathcal{C}(x)| > k] \le n e^{-\gamma k}.$$

If $\gamma k \geq (1+\varepsilon)\log n$ this tends to 0, so with high probability, $\mathcal{C}_1 \leq \frac{1-\varepsilon^2}{\varepsilon^2}\log n \leq \frac{1}{\varepsilon^2}\log n$. □

14.5 Super-Critical Erdös-Rényi

Lemma 14.5.1 For every $\delta > 0$ the following holds. $\mathbb{P}[|\mathcal{C}(1)| \geq \delta n] \geq 1 - q(X)$, where $X \sim \mathrm{Bin}(\lfloor(1-\delta)n\rfloor, p)$

Proof. Let $m = \lfloor(1-\delta)n\rfloor$. Let $T' = \inf\{t : N_t \leq m\}$. As long as $t < T'$, we have that $N_t > m$ and so there are at least m possible $y \in N_t$ to consider for the live x chosen from L_t. Thus, if we write $N_t = \{y_1 < y_2 < \cdots < y_m < y_{m+1} < \cdots y_k\}$ with $k = |N_t| = n - t - Y_t$, we can define

$$X_{t+1} := \sum_{j=1}^{m} \mathbf{1}_{\{x \sim y_j \text{ is open }\}} \sim \mathrm{Bin}(m,p),$$

conditional on $(L_s, D_s, N_s)_{s \leq t}$. Also by construction, $X_{t+1} \leq Z_{t+1} = Y_{t+1} - Y_t + 1$.

Thus the Galton-Watson process defined by $Y_t^* = Y_{t-1}^* + X_t - 1$ inductively satisfies $Y_t^* \leq Y_t$ for all $t < T'$.

Hence, if $Y_t = 0$ then either $Y_t^* = 0$ or $T' \leq t$. So for $T^* = \inf\{t : Y_t^* = 0\}$ we have that either $T' \leq T$ or $T^* \leq T$. Specifically, $T' \wedge T^* \leq T$.

Note that at time T', we have that $N_{T'} \leq m$ so $D_{T'} + Y_{T'} = n - N_{T'} \geq n - m \geq \delta n$. Since if $T' \leq T$ we have $|\mathcal{C}(1)| \geq D_{T'} + Y_{T'}$, we get that in any case $|\mathcal{C}(1)| \geq \min\{\delta n, T^*\}$.

Now, T^* is the size of a Galton-Watson tree of offspring distribution $X \sim \mathrm{Bin}(m,p)$, so

$$\mathbb{P}[T \geq \delta n] \geq \mathbb{P}[T^* > \delta n] \geq \mathbb{P}[T^* = \infty] \geq 1 - q(X).$$

□

Let us examine the process $(Y_t)_t$ more closely, in order to determine the distribution of the component size $|\mathcal{C}(1)| = T = \inf\{t : Y_t = 0\}$.

Lemma 14.5.2 Consider $Y_t = Y_{t-1} + Z_t - 1$ for $Z_t \sim \mathrm{Bin}(n-t+1-Y_{t-1}, p)$ are independent. Let this be defined for all $t \leq n-1$.

Then, for all $0 \leq t \leq n-1$,

$$|N_t| \sim \mathrm{Bin}(n-1, (1-p)^t).$$

Consequently,

$$Y_t \sim \mathrm{Bin}(n-1, 1-(1-p)^t) - t + 1.$$

Proof. Given $|N_t| = k$ we have that there are k neutral vertices to compare with our chosen live vertex, and each gets put into N_{t+1} with independent probability $(1-p)$. Thus,

$$|N_{t+1}| \;\Big|\; |N_t| = k \;\sim\; \mathrm{Bin}(k, 1-p).$$

It is now simple to verify that by induction, since $|N_0| = n-1 \sim \mathrm{Bin}(n-1, 1)$, we have that

$$|N_t| \sim \mathrm{Bin}(|N_{t-1}|, 1-p) \sim \mathrm{Bin}(n-1, (1-p)^t)$$

(see Exercise 14.1).

Now, $Y_t + t - 1 = n - |N_t| - |D_t| + t - 1 = n - 1 - |N_t|$. Thus, for all $0 \leq k \leq n-1$,

$$\mathbb{P}[Y_t + t - 1 = k] = \mathbb{P}[|N_t| = n-1-k] = \binom{n-1}{k}(1-p)^{t(n-1-k)}(1-(1-p)^t)^k.$$

$\square$

Exercise 14.1 Show that if $B \sim \mathrm{Bin}(n,p)$ and $X \mid B = k \sim \mathrm{Bin}(k,q)$ then $X \sim \mathrm{Bin}(n, pq)$. $\diamond\diamond\diamond$

We now carefully examine $(Y_t)_t$.

Lemma 14.5.3 For any $\varepsilon, \delta > 0$, there is n_0 such that for all $n > n_0$, if $p(n) = \frac{1+\varepsilon}{n}$, $k(n) \geq \frac{32}{\varepsilon^2}\log n$, and $q = q(\mathrm{Poi}(1+\varepsilon)$,

$$\mathbb{P}_{n,p(n)}[k(n) \leq T < (1-q)n \;\text{ or }\; T > (1+\delta)(1-q)n] \leq n^{-4}.$$

Proof. Fix $\varepsilon > 0$ small enough so that $\varepsilon < q(1+\varepsilon)$ where $q = q(\mathrm{Poi}(1+\varepsilon))$ is the extinction probability for Galton-Watson with $\mathrm{Poi}(1+\varepsilon)$ offspring. This is possible since as $\varepsilon \to 0$, $q \to 1$.

Assume that $p = p(n)$ such that $pn = 1+\varepsilon$.

We will make use of the inequalities $e^{-\xi} \leq 1 - \frac{\xi}{\xi+1}$ for any $0 \leq \xi \leq 1$, and $1-\xi \leq e^{-\xi}$ for any ξ. So with $p = \frac{\xi}{\xi+1} \iff \xi = \frac{p}{1-p}$, (as long as n is large enough $\xi \leq 1$) we have

$$1 - \exp(-pt) \leq 1 - (1-p)^t \leq 1 - \exp(-\tfrac{pt}{1-p}).$$

Let $t = \alpha n$ for $\alpha > 0$ such that $\alpha n \in \mathbb{N}$.

Then, when $\alpha < 1 - e^{-\alpha(1+\varepsilon)}$ we have that for $1 - \delta := \frac{\alpha}{1-e^{-\alpha(1+\varepsilon)}}$,

$$\begin{aligned}\mathbb{P}[Y_t \leq 0] &= \mathbb{P}[\mathrm{Bin}(n-1, 1-(1-p)^t) \leq t-1] \leq \mathbb{P}[\mathrm{Bin}(n-1, 1-e^{-\alpha(1+\varepsilon)}) \leq \alpha(n-1)] \\ &\leq \exp\left(-\tfrac{1}{4}\delta^2(n-1)(1-e^{-\alpha(1+\varepsilon)})\right) \leq \exp\left(-\tfrac{(1-e^{-\alpha(1+\varepsilon)}-\alpha)^2}{4\alpha(1+\varepsilon)}(n-1)\right).\end{aligned}$$

Now, if $\alpha(1+\varepsilon) \leq 1$ then

$$e^{-\alpha(1+\varepsilon)} \leq 1 - \alpha - \varepsilon\alpha + \alpha^2(1+\varepsilon)^2,$$

so

$$(1-e^{-\alpha(1+\varepsilon)} - \alpha)^2 \geq (\varepsilon\alpha + \alpha^2(1+\varepsilon)^2)^2 = \alpha^2 \cdot (\varepsilon + \alpha(1+\varepsilon)^2)^2 \geq \alpha^2 \cdot \varepsilon^2.$$

Now, the only solution in $(0,1)$ to $1 - q = 1 - e^{-(1+\varepsilon)(1-q)}$ is exactly $q = q(\mathrm{Poi}(1+\varepsilon))$, the extinction probability of a Galton-Watson process of offspring distribution $\mathrm{Poi}(1+\varepsilon)$. So $\alpha < 1 - e^{-\alpha(1+\varepsilon)}$ if and only if $\alpha < 1-q$.

Thus, we conclude that for all t such that $0 < t < n(1-q)$, since $t(1+\varepsilon) = n \cdot \alpha(1+\varepsilon) < n \cdot (1-q)(1+\varepsilon) \leq n$,

$$\mathbb{P}[Y_t = 0] \leq \exp\left(-\tfrac{\varepsilon^2}{4(1+\varepsilon)}t\right).$$

Taking $k < t = \lfloor n(1-q) \rfloor$,

$$\mathbb{P}[k \leq T < t] = \mathbb{P}[\exists\ k \leq m < t \ :\ Y_m = 0] \leq \sum_{m=k}^{t} \mathbb{P}[Y_m = 0] \leq n \cdot \exp\left(-\tfrac{\varepsilon^2}{4(1+\varepsilon)} \cdot k\right).$$

Since $1-q = 1-e^{-(1+\varepsilon)(1-q)}$, for any $\delta > 0$ we may choose $\varepsilon' > 0$ such that if $\alpha = (1-q)(1+\delta)$ then $\alpha = (1+\varepsilon')(1-e^{-(1+\varepsilon+\varepsilon')\alpha})$. Let $n_0 = n_0(\varepsilon')$ be large enough so that for all $n > n_0$, we have $\frac{p}{1-p} \leq 1+\varepsilon+\varepsilon'$. So

$$1-(1-p)^t \leq 1-\exp(-\tfrac{pt}{1-p}) \leq 1-e^{-t(1+\varepsilon+\varepsilon')/n}.$$

For $t \geq \alpha n$ we then have

$$\begin{aligned}\mathbb{P}[Y_t > 0] &= \mathbb{P}[\mathrm{Bin}(n-1, 1-(1-p)^t) > t-1] \leq \mathbb{P}[\mathrm{Bin}(n, 1-e^{-\alpha(1+\varepsilon+\varepsilon')}) \geq \alpha n] \\ &\leq \exp\left(-\tfrac{\varepsilon'^2}{4(1+\varepsilon')}\alpha n\right).\end{aligned}$$

Thus, for any $\delta > 0$ there exist $\varepsilon' > 0$ and $n_0 = n_0(\delta) > 0$ so that for all $n > n_0$,

$$\mathbb{P}_{n,p(n)}[T > \lceil (1+\delta)(1-q)n \rceil] \leq \exp\left(-\tfrac{\varepsilon'^2}{4(1+\varepsilon')}(1+\delta)(1-q)n\right).$$

Combined with the previous bound, we have that for any $\varepsilon, \delta > 0$, there is n_0 such that for all $n > n_0$, with $k \geq \frac{32}{\varepsilon^2}\log n$,

$$\mathbb{P}_{n,p(n)}[k \leq T < (1-q)n \ \text{ or } \ T > (1+\delta)(1-q)n] \leq n^{-4}.$$

□

We now turn to finally prove Theorem 14.3.2.

Proof of Theorem 14.3.2. Fix $\varepsilon, \delta > 0$, choose n large enough, and $p = p(n) = \frac{1+\varepsilon}{n}$.

Consider the following procedure to explore the components of $G(n,p)$. Start with $G_0 = G(n,p)$, $m_0 = n$, $C_0 = \emptyset$. Given G_j, m_j, C_j for $j \leq t$ define $G_{t+1} = G(n,p) \setminus (C_1 \cup C_2 \cup \cdots \cup C_t)$ and $m_{t+1} = |G_{t+1}| = n - |C_1| - \cdots - |C_t| = m_t - |C_t|$. Inductively G_{t+1} will have the distribution of $G(m_{t+1}, p)$. Choose any vertex of G_{t+1} and explore the component in G_{t+1} of that vertex. Let C_{t+1} be that component. Continue inductively.

Let $q = q(\mathrm{Poi}(1+\varepsilon)$. Let $q' = q(\mathrm{Bin}(\lfloor 1-\frac{\varepsilon}{2} \rfloor, p))$ (note that $(1-\frac{\varepsilon}{2})(1+\varepsilon) > 1$). Let $k = \lceil \frac{32}{\varepsilon^2}\log n \rceil$. Let r be an integer so that $(q')^r < n^{-4}$. So $r = O(\frac{\log n}{-\log q'})$. Choose n large enough so that $rkp << 1 - q(1+\varepsilon)$.

For all j, let S_j be the event that $|C_j| < k$ ('S' is for *small*). Let L_j be the event that $(1-q)m_j \leq |C_j| \leq (1+\delta)(1-q)m_j$ (*large*). Let B_j be the event that $S_j^c \cap L_j^c$ (*bad*).

Lemma 14.5.1 tells us that for $t \leq r$,

$$\mathbb{P}[S_t \bigm| S_1, \ldots, S_{t-1}] \leq q',$$

because $k << \varepsilon(n - rk)$ and on the event $S_1, \ldots, S_{t-1}$, we have that $m_t \geq n - tk \geq n - rk$.

Also, Lemma 14.5.3 tells us that

$$\mathbb{P}[B_t \bigm| G_j, C_j, j = 1, \ldots, t-1] \leq (m_t)^{-4}.$$

If $t \leq r$ and $S_1, \ldots, S_{t-1}$ then $m_t \geq n - rk$ so

$$\mathbb{P}[B_t \bigm| S_1, \ldots, S_{t-1}] \leq (n - rk)^{-4}.$$

Note that we have that either $S_1, \ldots, S_r$ or there exists $t \leq r$ with $S_1, \ldots, S_{t-1}, B_t$, or there exists $t \leq r$ with $S_1, \ldots, S_{t-1}, L_t$. So

$$\begin{aligned}\mathbb{P}[\exists\ t \leq r\ :\ S_1, \ldots, S_{t-1}, L_t] &\geq 1 - \mathbb{P}[S_1, \ldots, S_r] - \mathbb{P}[\exists\ t \leq r\ :\ S_1, \ldots, S_{t-1}, B_t] \\ &\geq 1 - (q')^r - r(n - rk)^{-4}.\end{aligned}$$

Since for $t \leq r$, $S_1, \ldots, S_{t-1}$ implies that $m_t > n - rk$, we get that with high probability there exists $t \leq r$ such that $m_t > n - rk$ and $|C_t| \geq (1-q)m_t \geq (1-q)(n-rk)$. So $m_{t+1} \leq qn + (1-q)rk$. and $m_{t+1}p \leq q(1+\varepsilon) + rkp <<$ 1. This implies that G_{t+1} is sub critical, and has only logarithmic size components, with high probability. $\square$