ENEE620. Midterm examination, 4/29/2021. Instructor: A. Barg

- Please submit your work to ELMS Assignments as a single PDF file by Friday, 4/30/21, 10:00am EDT.
- Each problem is 10 points. Max score=50 points
- Your answers should be justified. Giving just the answer may result in no credit for the problem.
- Please pay attention to the writing. You may lose points if your paper is difficult to read.

Problem 1.

Let $(Y_i)_{i\ge 1}$ be independent random digits, i.e., independent uniform RVs on $\{0,1,\dots,9\}$. Let $X_n, n\ge 1$ be the number of different digits among $Y_1,Y_2,\dots,Y_n$.

(a) Show that the sequence $(X_n)_n$ forms a Markov chain and find its matrix of transition probabilities.

(b) Find the probability mass function of X_4 (please give the answer as a set of numbers).

(c) Classify the states into recurrent and transient. Justify your answers.

SOLUTION: (a) The process has 11 states, and the transition probabilities are as follows:

$$p_{ij}=\begin{cases}0, & j<i\\ i/10, & j=i\\ 1-i/10, & j=i+1\\ 0, & j>i+1\end{cases}$$

(b) There are the following possibilities for the first 4 steps of the process: $(1111),(1112),(1122),(1222),(1123),(1223),(1233),(1234)$, yielding the values $X_4=1,2,3,4$. Performing the calculations and writing $P(i)=\Pr(X_4=i)$, we obtain $P(1)=0.001, P(2)=0.063, P(3)=0.432, P(4)=0.504$.

(c) It is intuitively clear that after sufficiently many steps all the digits will appear, and since $p_{ii}=1$ for $i=10$, the state 10 is (recurrent and) absorbing, while all the other states are transient. To justify this, consider the probability $p_n(i)$ of staying in state i for n steps, $p_n(i)=(i/10)^n$. With probability 1 the process will leave i after some number of steps, moving to $i+1$. By the same argument, with probability 1 after some number of steps the process reaches state 10, which is a recurrent, absorbing state. The remaining states are therefore transient.

Problem 2.

(a) RVs $(X_n)_{n\ge 0}$ form a Markov chain with the matrix of transition probabilities P. Show that the RVs X_n are independent if and only if all the rows of P are identical (Hint: calculate the joint distribution $P(X_0=i_0,\dots,X_n=i_n)$)

(b) Let $(X_n)_{n\ge 0}$ be a sequence of independent RSs. Does the sequence $Y_n=X_{n-1}+X_n, n\ge 1$ form a (first-order) Markov chain? Justify your answer.

SOLUTION: (a) Assume that the RVs are independent, then $p_{ij}=P(X_n=j|X_{n-1}=i)=P(X_n=i)$, and thus this probability does not depend on i, so the rows are identical. Conversely, assume that $p_{ij}=a_j$ for all i. Now consider

$$P(X_0=i_0,\dots,X_n=i_n)=P(X_0=i_0)\prod_{m=1}^n P(X_m=i_m|X_{m-1}=i_{m-1})=P(X_0=i_0)\prod_{j=1}^n a_j.$$

Next,

$$P(X_1=i_1)=\sum_{i_0\in S}P(X_1=i_1|X_0=i_0)P(X_0=i_0)=a_{i_1}\sum_{i_0\in S}P(X_0=i_0)=a_{i_1}$$

and similarly, $P(X_n=i_n)=a_{i_n}$. This implies that the joint probability $P(X_0=i_0,\dots,X_n=i_n)$ for any n breaks into a product, establishing the independence.

(b) No, generally it doesn't. For instance, if the X_n's are iid Bernoulli with $P(1)=p, P(0)=1-p, p\ne 1/2$, then

$$P(Y_3=1|Y_2=1,Y_1=0)=\frac{P(X_0=0,X_1=0,X_2=1,X_3=0)}{P(X_0=0,X_1=0,X_2=1)}=1-p$$

but

$$P(Y_3 = 1|Y_2 = 1) = \frac{P(010) + P(101)}{P(01) + P(10)} = \frac{1}{2}.$$

Problem 3.

(a) Let M, a be a pair of nonnegative RVs with some joint cdf F, and let an independent RV θ be uniform on $[0, 2\pi)$. Prove that the random process

$$X(t) = M\cos(at + \theta), \quad t \in \mathbb{R}$$

is stationary (in the general sense, not the wide sense). The RVs M, a are not assumed to be independent or Gaussian.

(b) Let a, b be independent Bernoulli RVs that take values ± 1 with equal probabilities. Show that

$$X(t) = a\cos mt + b\sin mt,$$

where $t \in \mathbb{R}$ and m is some constant, is WSS (wide-sense stationary).

SOLUTION: (a) Let $B_1, \dots, B_n$ be any Borel sets, and consider

$$P(M\cos(a(t_1 + h) + \theta) \in B_1, \dots, M\cos(a(t_n + h) + \theta) \in B_n).$$

Note that the argument of the cosines is shifted by Mh and reduced $\mod 2\pi$, and θ is uniform on $[0, 2\pi)$, so this probability is the same as $P(M\cos(at_1 + \theta) \in B_1, \dots, M\cos(at_n + \theta) \in B_n)$, proving stationarity. Formally, we may consider the integral

$$\begin{aligned} &P(M\cos(a(t_1 + h) + \theta) \in B_1, \dots, M\cos(a(t_n + h) + \theta) \in B_n) \\ &\qquad = \int_0^\infty \int_0^\infty P(x\cos(y(t_1 + h) + \theta) \in B_1, \dots, x\cos(y(t_n + h) + \theta) \in B_n) F(dx, dy) \end{aligned}$$

arriving at the same conclusion.

(b) We have $EX(t) = 0$, and

$$\begin{aligned} E[X(t)X(s)] &= E[(a\cos mt + b\sin mt)(a\cos ms + b\sin ms)] \\ &= \cos mt\cos ms + \sin mt\sin ms = \cos(m(t - s)) \end{aligned}$$

as required. Note that this process is easily seen to be not stationary in the general sense.

Problem 4.

An $M/M/s$ queueing system does not have a buffer to hold pending requests. An incoming request is assigned to one of the available (idle) servers if there are one or more idle servers at the time of arrival. If there are no available servers, the request leaves the system and never returns. Assuming that the arrival rate is λ and the service rate of each of the servers is μ, find the stationary distribution of the system.

SOLUTION: The process has $s+1$ states corresponding to $0, 1, \dots, s$ customers in the system. The generator matrix has the form

$$Q_{ij} = \begin{cases} -\lambda & i = j = 0 \\ \lambda & j = i + 1 < s \\ i\mu & j = i - 1 \\ -s\mu & i = j = s \\ -\lambda - i\mu & 0 < i = j < s \end{cases}$$

This is a birth-death chain, and the relation $\pi Q = 0$ yields the "flow equations" $\pi_i q_{i,i+1} = \pi_{i+1} q_{i+1,i}$. This in turn gives

$$\pi_i = \pi_0 \prod_{j=0}^{i-1} \Big(\frac{\lambda}{(i+1)\mu}\Big),$$

and adding these relations, we find

$$\pi_0 = \frac{1}{\sum_{i=0}^{s}(\frac{\lambda}{\mu})^i/i!}.$$

Problem 5.

Let $W_n, n \geq 1$ be i.i.d. Gaussian RVs $\sim \mathcal{N}(0, \sigma^2)$. Let $X_0 = 0$ and define a Gauss-Markov process

$$X_n = aX_{n-1} + W_n, \quad n \geq 1,$$

where $a \in (-1, 1)$ is a real number.

(a) Show that $X_n, n \geq 1$ is a zero-mean discrete-time Gaussian process.

(b) Prove that the covariance function (matrix) of the process X_n has the form

$$E[X_n X_{n+k}] = \frac{\sigma^2(1-a^{2n})a^k}{1-a^2}.$$

(Hint: use induction on k)

SOLUTION: (a) The definition tells us that each X_n is a linear combination of independent zero-mean Gaussian RVs, and thus each X_n is itself zero-mean Gaussian. Thus, the RVs $X_n, n \geq 1$ form a Gaussian sequence, or, in other words, $(X_n)_n$ is a discrete-time zero-mean Gaussian process.

(b) Writing the definition of X_n recursively, we see that

$$X_n = \sum_{i=1}^{n} a^{n-1} W_i.$$

Let us check the induction base:

$$EX_n^2 = E\Big[\sum_{i=1}^{n} a^{n-1} W_i \sum_{i'=1}^{n} a^{n-1} W_{i'}\Big] = \sum_{i,i'} a^{2n-i-i'} E[W_i W_{i'}]$$

$$= \sum_{i=1}^{n} a^{2(n-i)}\sigma^2 = \sigma^2 \sum_{i=0}^{n-1} a^{2i} = \sigma^2 \frac{1-a^{2n}}{1-a^2}.$$

Assume that $E(X_n X_{n+k-1} = \frac{\sigma^2(1-a^{2n})a^{k-1}}{1-a^2}$ and let us make the induction step:

$$E[X_n X_{n+k}] = E[X_n(aX_{n+k-1} + W_{n+k})] = aE(X_n X_{n+k-1}) + E(X_n W_{n+k})$$

Since X_n and W_{n+k} are independent, the last term is 0, and we proceed:

$$E[X_n X_{n+k}] = a * \frac{\sigma^2(1-a^{2n})a^{k-1}}{1-a^2},$$

as required.