ENEE620. Midterm examination, 3/25/2021. Instructor: A. Barg

- Please submit your work to ELMS Assignments as a single PDF file by 3/26/21, 10:00am EDT.
- Each problem is 10 points. Max score=50 points
- Your answers should be justified. Giving just the answer may result in no credit for the problem.
- Please pay attention to the writing. You may lose points if your paper is difficult to read.

Problem 1. Given a probability space $(\Omega, \mathcal{F}, P)$.

(1) Show that for any sequence of events $(A_n)_n$

$$P(\liminf A_n) \le \liminf P(A_n) \le \limsup P(A_n) \le P(\limsup A_n).$$

(2) Describe the σ-algebra generated by (i) all events $A \subset \mathcal{F}$ with $P(A) = 0$; (ii) all events $A \subset \mathcal{F}$ with $P(A) = 1$ (explicitly describe all events that form this σ-algebra).

(3) Now assume that $\Omega = \mathbb{R}$, $\mathcal{F} = \mathcal{B}(\mathbb{R})$. Describe the σ-algebra generated by the RV $X = \cos \omega$.

SOLUTION: (1) We write

$$\begin{aligned}P(\liminf A_n) &= P(\cup_{n=1}^\infty \cap_{m=n}^\infty A_m) = \lim_{n\to\infty} P(\cap_{m=n}^\infty A_m)\\ &\le \liminf P(A_n) \le \limsup P(A_n) = \lim_{n\to\infty} P(\cup_{m=n}^\infty A_m)\\ &= P(\cap_{n=1}^\infty \cup_{m=n}^\infty A_m) = P(\limsup A_n).\end{aligned}$$

(2) If $P(A) = 0$, then $P(A^c) = 1$. A union of any number of events with probability 1 has probability 1. Further, a (countable) union of disjoint events of probability zero has probability zero, and this is all the more true if the events are not disjoint. Thus, the answer in both cases (i) and (ii) is "all events of probability 0 and 1".

(3) It is formed of all sets of the form $\cup_{n=-\infty}^\infty (B + 2\pi n)$, where $B \in \mathcal{B}([-\pi, \pi])$ is a Borel set in the σ-algebra of Borel sets on the segment $[-\pi, \pi]$.

Problem 2.

(1) Show that for any sequence of RVs $(X_n)_n$,

$$X_n \xrightarrow{p} 0 \text{ is equivalent to } \frac{X_n^2}{1+X_n^2} \xrightarrow{p} 0.$$

(2) Let $(X_n)_n$ be independent random variables with finite expectations and let $Y_n = \prod_{i=1}^n X_i, n \ge 1$.

(a) Show that if $1 > E|X_1| = E|X_2| = \dots$, then $Y_n \xrightarrow{p} 0$.

(b) Suppose in addition that X_n are identically distributed with common expectation EX. Is the converse of the statement in part (a) true? In other words, does $Y_n \xrightarrow{p} 0$ imply that $EX < 1$? (prove or give a counterexample).

SOLUTION:
(1) Note that $\frac{X_n^2(\omega)}{1+X_n^2(\omega)} \le \epsilon^2$ if and only if $|X_n(\omega)| \le \frac{\epsilon}{\sqrt{1-\epsilon^2}}$.
(2) Markov's inequality suffices: for $n \to \infty, \epsilon > 0$ using $E(X_1) < 1$

$$P(|Y_n| > \epsilon) < \frac{E|Y_n|}{\epsilon} = \frac{(E|X_1|)^n}{\epsilon} \to 0.$$

At the same time, let X_n be Bernoulli with $P(X_n = 0) = p, p \in (0,1)$ and $P(X_n = a) = 1 - p$. Then $P(Y_n = 0) = 1 - (1-p)^n$ and $Y_n \xrightarrow{p} 0$, while $EX_n = a(1-p)$, which can be made > 1 by taking $a > 1/(1-p)$. Thus, the converse statement is false.

Problem 3. Given a sequence $(X_n)_n$ of i.i.d. Gaussian $(0, 1)$ RVs.

(1) Let $A_n = \{\omega : X_1(\omega) + \cdots + X_n(\omega) > a\sqrt{2n \ln n}\}, n \ge 1$, where $a > 1$ is a fixed number. Show that $P(A_n \text{ i.o.}) = 0$.

(2) Show that

$$P\Big(\limsup \frac{X_n}{\sqrt{2\ln n}} = 1\Big) = 1.$$

SOLUTION: (Borel-Cantelli lemmas)

(1) Let $Y_n = X_1 + \cdots + X_n$, then $Y_n \sim \mathcal{N}(0, n)$. Using the inequality given in HW1, Problem 3, we have

$$P(A_n) = P(Y_n > a\sqrt{2n\ln n}) = P(Y_n n^{-1/2} > a\sqrt{2\ln n}) = 1 - \Phi(a\sqrt{2\ln n})$$

$$\le ca\sqrt{2\ln n} \times \Big(\frac{1}{\sqrt{2\pi}} e^{-a^2 \ln n}\Big) = C' n^{-a^2}\sqrt{2\ln n}.$$

Since $a > 1$, we have $\sum_{n=1}^{\infty} P(A_n) < \infty$, and this $P(A_n \text{ i.o.}) = 0$, as required.

(2) We again rely on the estimate $1-\Phi(x) \le cx\phi(x)$. As above, letting $\epsilon > 0$ and $B_n(\epsilon) = \{X_n > \sqrt{2\ln n}(1+\epsilon)\}$, we find

$$P(B_n(\epsilon)) \le C\sqrt{2\ln n} n^{-1+\epsilon},$$

and this implies that $P(B_n(\epsilon) \text{ i.o.}) = 0$. Likewise, the events $B_n(-\epsilon)$ are independent, and we can show that $P(B_n(-\epsilon) \text{ i.o.}) = 1$ (using an estimate of the form $1 - \Phi(x) \ge O(1/x)\phi(x)$). These two results together imply the needed claim.

Problem 4.

(1) Consider a renewal process with interarrival pdf given by $f(t) = \frac{1}{2}(e^{-t} + 3e^{-3t})$. Show that the renewal function has the form $m(t) = \frac{3t}{2} - \frac{1}{4}(e^{-2t} - 1)$.

(2) Now compute the renewal function $m(t)$ when interarrivals are distributed according to a general mixture of two exponentials:

$$f(t) = p\lambda_1 e^{-\lambda_1 t} + (1-p)\lambda_2 e^{-\lambda_2 t}.$$

SOLUTION: Eq. (G) from HW3 tells us that

$$\int_0^{\infty} e^{-st} m'(t) dt = \frac{G_X(s)}{1 - G_X(s)}.$$

(1) Let us compute the Laplace transform of the distribution of the interarrival times:

$$G_X(s) = \int_0^{\infty} \frac{1}{2}(e^{-t} + e^{-3t}) e^{-st} dt = \frac{1}{2}\Big(\frac{1}{s+1} + \frac{3}{s+3}\Big) = \frac{2s+3}{(s+1)(s+3)}.$$

Denoting the Laplace transform of a function $g(t)$ by $L(g(s)$ and using the above equation, we find

$$L_{m'}(s) = \frac{G_X(s)}{1 - G_X(s)} = \frac{2s+3}{(s+1)(s+3) - (2s+3)} = \frac{2s+3}{s(s+2)}.$$

The inverse Laplace transform now gives

$$m'(t) = 2e^{-2t} + \frac{3}{2}(1 - e^{-2t}), \quad t > 0.$$

Integrating,

$$m(t) = \int_0^t (2e^{-2u} + \frac{3}{2}(1 - e^{-2u}))du = \frac{3t}{2} - \frac{1}{4}e^{-2t} + C,$$

and since $m(t) = 0$, we find $C = \frac{1}{4}$, as required.

(2) The general version of this question retraces the steps in Part (1). To save on writing, let us denote $\bar{p} := 1 - p$.

$$G_X(s) = p\lambda_1 \frac{1}{s+\lambda_1} + \bar{p}\lambda_2 \frac{1}{s+\lambda_2} = \frac{(p\lambda_1 + \bar{p}\lambda_2)s + \lambda_1\lambda_2}{(s+\lambda_1)(s+\lambda_2)}.$$

Hence

$$L_{m'}(s) = \frac{G_X(s)}{1 - G_X(s)} = \frac{(p\lambda_1 + \bar{p}\lambda_2)s + \lambda_1\lambda_2}{s(s + \bar{p}\lambda_1 + p\lambda_2)} = \frac{p\lambda_1 + \bar{p}\lambda_2}{s + \bar{p}\lambda_1 + p\lambda_2} + \frac{\lambda_1\lambda_2}{s(s + \bar{p}\lambda_1 + p\lambda_2)}$$

The inverse Laplace transform now gives

$$m'(t) = (p\lambda_1 + \bar{p}\lambda_2)e^{-(\bar{p}\lambda_1 + p\lambda_2)t} + \frac{\lambda_1\lambda_2}{\bar{p}\lambda_1 + p\lambda_2}(1 - e^{-(\bar{p}\lambda_1 + p\lambda_2)t}).$$

Integrate and use $m(0) = 0$ to compute

$$m(t) = \frac{\lambda_1\lambda_2}{\bar{p}\lambda_1 + p\lambda_2}t - \frac{p\bar{p}(\lambda_1 - \lambda_2)^2}{(\bar{p}\lambda_1 + p\lambda_2)^2}(e^{-(\bar{p}\lambda_1 + p\lambda_2)t} - 1).$$

Problem 5.

We consider a Poisson proces $N(t)$ with arrival rate λ. The nth arrival of the process constitutes a random payment of size $Y_n, n \geq 1$. Suppose that the sizes of the payments are i.i.d. RVs with characteristic function $\phi_Y(t) = \lambda/(\lambda - it), \lambda > 0$. Denote by $S(t) = Y_1 + \cdots + Y_{N(t)}$ the total amount of payments at time t.

Find the mean and variance of $S(t)$.

SOLUTION:

This is a compound Poisson process (a.k.a. Poisson process with rewards). Plainly, $Y \sim \text{Exp}(\lambda)$, so $EY = \frac{1}{\lambda}$, $\text{Var}(Y) = \frac{1}{\lambda^2}$, and $EY^2 = \frac{2}{\lambda^2}$.

The mean and variance of S as functions of t are easily found by conditioning on the random number of arrivals and Wald's identity:

$$S(t) = \sum_{i=1}^{N(t)} Y_i,$$

so

$$ES(t) = EN(t)EY = (\lambda t) \cdot \frac{1}{\lambda} = t.$$

Similarly, given $N(t) = n$, we find $\text{Var}\,S(t) = n\text{Var}(Y)$, and

$$E(S^2(t)|N(t) = n) = n\text{Var}(Y) + (nE(Y))^2,$$

$$\begin{aligned} E(S(t)^2) &= \sum_{n=0}^{\infty} E(S^2(t)|N(t) = n)P(N(t) = n) = \sum_{n=0}^{\infty}(n\text{Var}(Y) + n^2(EY)^2)P(N(t) = n) \\ &= EN(t)\text{Var}(Y) + E(N(t)^2)(EY)^2, \end{aligned}$$

and

$$\begin{aligned} \text{Var}(S(t)) &= ES(t)^2 - (ES(t))^2 = EN(t)\text{Var}(Y) + E(N(t)^2)(EY)^2 - (EN(t)EY)^2 \\ &= EN(t)\text{Var}(Y) + \text{Var}(N(t))(EY)^2. \end{aligned}$$

Since for the Poisson process, $EN(t) = \text{Var}(N(t)) = \lambda t$, we otbain $\text{Var}(S(t)) = \lambda t \cdot EY^2 = \frac{2t}{\lambda}$.