ENEE 620 Midterm 1, Solutions

1. We have

$$P(X\in B) \geq P((X\in B)\cap(X=Y)) = P((Y\in B)\cap(X=Y))$$
$$= 1 - P((Y\notin B)\cup(X\neq Y)) \geq P(Y\in B) - P(X\neq Y) \qquad (1)$$

Further

$$P(X\in B) = P((X\in B)\cap(X=Y)) + P((X\in B)\cap(X\neq Y))$$
$$\leq P(Y\in B) + P(X\neq Y) \qquad (2)$$

Taken together

$$-P(X\neq Y) \overset{(1)}{\leq} P(X\in B) - P(Y\in B) \overset{(2)}{\leq} P(X\neq Y), \text{ as claimed}$$

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2. (a) $P(X_n\neq 0) = P(Y\leq \frac{1}{n}) = \frac{1}{n} \to 0$

$\therefore \lim_{n\to\infty} P(X_n \geq \varepsilon) = 0$ for arbitrarily small $\varepsilon > 0$

This implies $X_n \xrightarrow{P} 0$

Now let $A_n = \{X_n\neq 0\}$, then $\{\lim_{n\to\infty} X_n \neq 0\} = \{\bigcap_{m\geq 1}\bigcup_{n\geq m} A_n\}$; $A_n \supset A_{n+1}$ (all n)

$$P\Big(\bigcap_{m=0}^{\infty}\bigcup_{n=m}^{\infty} A_n\Big) = \lim_{m\to\infty} P\Big(\bigcup_{n=m}^{\infty} A_n\Big) = \lim_{m\to\infty} P(A_m) = 0 \qquad \therefore X_n \xrightarrow{a.s.} 0$$

Now notice that $P(X_n^2 = n) = \frac{1}{n}$, so

$EX_n^2 = n\cdot\frac{1}{n} = 1$, thus $X_n \overset{m.s.}{\not\to} 0$

(X_n does not have an m.s. limit)

(b) Now $X_n = \sqrt{n}\,\mathbb{1}_{\{Y_n\leq \frac{1}{n}\}}$

$P(X_n\neq 0) = P(Y_n\leq \frac{1}{n}) \to 0$, thus $X_n \xrightarrow{P} 0$

To address a.s. convergence, consider independent events

$A_n = \{X_n\neq 0\}$; $P(A_n) = \frac{1}{n}$ and $\sum_n P(A_n) = \infty$

Borel-Cantelli tells us that $P(A_n \text{ i.o.}) = 1$, so

$P(\lim_{n\to\infty} X_n = 0) = 0$. $X_n \overset{a.s.}{\not\to} 0$

Finally, $EX_n^2 = nP(Y_n\leq\frac{1}{n}) = 1$, so $X_n \overset{m.s.}{\not\to} 0$

3. Consider the independent events $X_n > \lambda_n$; if $\sum_{n=1}^{\infty}(1-F(\lambda_n)) = \infty$, then $P(\{X_n > \lambda_n\} \text{ i.o.}) = 1$.

Since $\{X_n > \lambda_n\} \subset A_n$, in this case also $P(A_n \text{ i.o.}) = 1$

Now suppose that $\sum_{n=1}^{\infty}(1-F(\lambda_n)) < \infty$, i.e. $P(\{X_n > \lambda_n\} \text{ i.o.}) = 0$

Assume in addition that the sequence $(\lambda_n)_{n \geq 1}$ is monotone increasing, $\lambda_n \uparrow \infty$.

We need to show that if $X_n \leq \lambda_n$ a.s., also $\max_{1 \leq m \leq n} X_m \leq \lambda_n$ a.s.

$1 = P(X_n > \lambda_n \text{ finitely many times})$

$= P(\bigcap_{N \geq n} \{X_n > \lambda_N\} \text{ finitely many times})$ (by monotonicity)

$= P(\bigcap_{n=1}^{N} \{X_n > \lambda_N\} \text{ finitely many times}) = P(A_N^c \text{ i.o.})$

$-\ln(1-(1-p))$
$= -\ln p$

4. (a) $P_X(n) = p(1-p)^n, \quad n = 0, 1, \ldots$

recognize the circled sum as
$-\ln(1-(1-p)) = -\ln p$

$$E\left(\frac{1}{1+X}\right) = p\sum_{n=0}^{\infty}\frac{(1-p)^n}{n+1} = \frac{p}{1-p}\sum_{n=1}^{\infty}\frac{(1-p)^n}{n} = \frac{p}{1-p}\ln\frac{1}{p}.$$

(b) $$P(X_n \leq nt) = 1-\left(1-\frac{\lambda}{\lambda+n}\right)^{nt} = 1-\left(1-\frac{\lambda}{\lambda+n}\right)^{\frac{\lambda+n}{\lambda}\cdot\frac{\lambda}{\lambda+n}\cdot nt}$$

$$\xrightarrow[n\to\infty]{} \underbrace{1-e^{-\lambda t}}_{\text{CDF Exp}(\lambda)}, \text{ i.e., } \frac{X_n}{n} \xrightarrow{d} \text{Exp}(\lambda)$$

5. (a) $P(\min(X,Y)\le n) = 1 - P(X>n \cap Y>n) = 1 - \left(\sum_{m=n+1}^{\infty} 2^{-m}\right)^2$

$= 1 - \left(\frac{2^{-n-1}}{1/2}\right)^2 = 1 - 4^{-n}.$

(b) $P(X=Y) = \sum_{n=1}^{\infty} P(X=n)^2 = \sum_{n=1}^{\infty} 2^{-2n} = \frac{1/4}{3/4} = \frac{1}{3}$

(c) $P(X>Y) = \sum_{n=1}^{\infty} P(Y=n) \sum_{m=n+1}^{\infty} P(X=m) = \sum_{n=1}^{\infty} 2^{-n} \sum_{m=n+1}^{\infty} 2^{-m} = \sum_{n=1} 2^{-n} \frac{2^{-n-1}}{1/2}$

$= \sum_{n=1}^{\infty} 2^{-2n} = 1/3$

(d) $P(X|Y) = \sum_{n=1}^{\infty} P(X=n) \sum_{m=1} P(Y=mn) = \sum_{n=1}^{\infty} 2^{-n} \sum_{m=1}^{\infty} (2^{-n})^m$

$= \sum_{n=1}^{\infty} 2^{-n} \frac{2^{-n}}{1-2^{-n}} = \sum_{n=1}^{\infty} \frac{1}{2^n(2^n-1)}$ (no closed-form expression :( )

(e) $P(X\ge kY) = \sum_{n=1}^{\infty} P(Y=n)\, P(X\ge kn) = \sum_{n=1}^{\infty} 2^{-n} \sum_{m=kn}^{\infty} 2^{-m}$

$= \sum_{n=1}^{\infty} 2^{-n} \frac{2^{-kn}}{1-1/2} = 2\sum_{n=1}^{\infty} 2^{-(k+1)n} = 2\frac{2^{-(k+1)}}{1-2^{-(k+1)}} = \frac{2}{2^{k+1}-1}$