Mathematical prerequisites

Elements of notation

$\mathbb{R}$ reals

$\mathbb{Q}$ rationals $\frac{m}{n}$

$\mathbb{C}$ complex numbers

$\mathbb{N} := \{1,2,\dots\}$

$\mathbb{N}_0 := \mathbb{N} \cup \{0\}$

$\mathbb{Z} = \{\dots -2,-1,0,1,2,\dots\}$

$\mathbb{Z}^+ = \mathbb{N}_0$

$\mathbb{R}^2 = \{(x,y),\ x,y \in \mathbb{R}\}$

$\mathbb{R}^n$ = n-dim Euclidean space

Intervals of the real line $\mathbb{R}$

$[a,b]$ $\quad a \le x \le b$

$(a,b]$ $\quad a < x \le b$

$[a,b)$ $\quad a \le x < b$

(a,b) $\quad a < x < b$

Set theoretic notation

$A \cup B$; $A \cap B$; A^c ; $\bigcup_{n=1}^{\infty} A_n$

Ω = universal set (Sample space) $\qquad A \subset \Omega$; $A^c = \Omega \setminus A$

$$\mathbb{1}_A(\omega) = \begin{cases} 1 & \omega \in A \\ 0 & \omega \notin A \end{cases} \qquad \omega \in \Omega$$

indicator function of A

Countable and uncountable sets

A set A is countable (countably infinite) if there is a one-to-one map between A and $\mathbb{N}$. The sets $\mathbb{N}_0$, $\mathbb{Z}$, $\mathbb{Q}$ are countable; the set of all infinite sequences of coin tosses is not. The sets $\mathbb{R}$, {all points on $[0,1]$} are uncountable.

Key concepts

1) $\lim_{n\to\infty} x_n = x$

This means that for any neighborhood of x, $N_\varepsilon(x)$, the sequence (x_n) stays

$$N_\varepsilon(x) = \{y \in \mathbb{R} : |x-y| \le \varepsilon\}$$

within the neighborhood for all $n \geq n_0$

where $n_0 = n_0(\varepsilon)$

Rephrasing again, the sequence can be outside $N_\varepsilon(x)$ only finitely many times.

$\forall \varepsilon > 0\ \exists n_0 = n_0(\varepsilon)$ s.t. $\forall n \geq n_0$, $x_n \in N_\varepsilon(x)$

The limit can be ∞, meaning that x_n increases indefinitely, i.e., $\forall N \in \mathbb{N}$ $\exists n_0 = n_0(N)$ s.t.

$$x_n > N \quad \text{for all } n \geq n_0.$$

It is also possible that

$\lim_{n\to\infty} x_n$ does not exist.

2) $\sum_{n=1}^{\infty} x_n = x \overset{\text{Def}}{\Longleftrightarrow} \lim_{N\to\infty} \sum_{n=1}^{N} x_n = x$

If the series converges, then

$$\lim_{m\to\infty} \sum_{n=m}^{\infty} x_n = 0$$

If $\sum_{n=1}^{\infty} x_n = \infty$, we say that the series diverges

If no limit, then we also say "diverges"

$$\sum_{n=1}^{\infty} \frac{1}{n^s} = \begin{cases} \infty & s \leq 1 \\ < \infty & s > 1 \end{cases} \qquad s = 2 \quad = \frac{\pi^2}{6}$$

3) **Taylor series**

$$\ln(1 \pm x) = \pm x - \frac{x^2}{2} \pm \frac{x^3}{3} - \frac{x^4}{4} \pm \frac{x^5}{5} - \dots \qquad |x| < 1$$

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots, \quad x \in \mathbb{R}.$$

4) $\limsup_{n\to\infty} x_n = \inf_{m \geq 1} \sup_{n \geq m} x_n$

$\liminf_{n\to\infty} x_n = \sup_{m \geq 1} \inf_{n \geq m} x_n$

5) **Continuous functions**

A function $f(x)$ is continuous at x_0 if for any sequence $x_n \xrightarrow{n\to\infty} x_0$ the sequence $f(x_n)$ converges to $f(x_0)$.

6) **Matrices**

An $n\times n$ matrix M is symmetric if $M = M^T$.

A matrix M is orthogonal if $M^{-1} = M^T$

$\det(A)$ = determinant of A

$\det(AB) = \det(A)\det(B)$

Eigenvalues ; left and right eigenvectors