Problem 1:

Problem 1. Consider a standard Galton-Watson branching process $(X_n)_{n\geq 0}$ with $X_0 = 1$ and $X_{n+1} = \sum_{k=1}^{X_n} Z_{n+1}^{(k)}$, where $(Z_n^{(k)}, n \geq 1, k \geq 1)$ is a collection of iid RVs with finite expectation μ and variance σ^2, taking values in $\mathbb{N}_0$.

(a) Prove that $M_n := X_n/\mu^n$ forms a martingale with respect to the natural filtration $(\mathcal{F}_n)_n$ defined by $\mathcal{F}_n = \sigma(X_0, X_1, \ldots, X_n)$.

(b) Show that $E(X_{n+1}^2|\mathcal{F}_n) = \mu^2 X_n^2 + \sigma^2 X_n$.

(c) Show that M_n is bounded in L^2 (i.e., $\sup_{n\geq 1} EM_n^2 < \infty$) if and only if $\mu > 1$.

(d) Show that for $\mu > 1$, $\mathrm{Var}(M_\infty) = \sigma^2/(\mu(\mu-1))$.

(a) $$E\left[\frac{X_{n+1}}{\mu^{n+1}} \,\Big|\, \mathcal{F}_n\right] = \frac{1}{\mu^{n+1}} E\left[\sum_{k=1}^{X_n} Z_{n+1}^{(k)} \,\Big|\, \mathcal{F}_n\right] = \frac{1}{\mu^{n+1}} \mu X_n = \frac{X_n}{\mu^n}$$

Let us check that M_n is integrable:

$EX_0 = 1$; $EX_n = \mu^n, n \geq 1$, so $E|M_n| = 1$ for all $n \geq 1$

This proves the martingale claim.

(b) $$E\left[X_{n+1}^2 \,|\, \mathcal{F}_n\right] = E\left[\sum_{i,j=1}^{X_n} Z_{n+1}^{(i)} Z_{n+1}^{(j)} \,\Big|\, \mathcal{F}_n\right]$$

$$= E\left[\sum_{i=1}^{n} \left(Z_{n+1}^{(i)}\right)^2 \,\Big|\, \mathcal{F}_n\right] + E\left[\sum_{i\neq j} Z_{n+1}^{(i)} Z_{n+1}^{(j)} \,\Big|\, \mathcal{F}_n\right]$$

$$= (\sigma^2 + \mu^2) X_n^2 + \mu^2 (X_n^2 - X_n)$$

$Z_{n+1}^{(i)} \perp\!\!\!\perp Z_{n+1}^{(j)}$ for $i \neq j$

$X_n^2 - X_n$ terms

$$= \mu^2 X_n^2 + \sigma^2 X_n$$

(c) We compute

$$E M_n^2 = \frac{1}{\mu^{2n}} E X_n^2 = \frac{1}{\mu^{2n}} E\left[E\left[X_n^2 \,|\, \mathcal{F}_{n-1}\right]\right]$$

$$\overset{\text{Pt.(b)}}{=} \frac{1}{\mu^{2n}} E\left[\mu^2 X_{n-1}^2 + \sigma^2 X_{n-1}\right]$$

$$= E M_{n-1}^2 + \frac{\sigma^2}{\mu^{n+1}} (\underbrace{E M_{n-1}}_{=1}) = \ldots = 1 + \sum_{i=1}^{n} \frac{\sigma^2}{\mu^{n+1}}$$

(d) Since the process $(M_n)_n$ is L_1-uniformly bounded, $M_n \xrightarrow{a.s.} M_\infty$ and $EM_n^2 \to EM_\infty^2$.

$$\therefore EM_\infty^2 = 1 + \sum_{i=1}^{\infty} \frac{\sigma^2}{\mu^{i+1}} = 1 + \frac{\sigma^2}{\mu(\mu-1)} \quad \text{if } \mu > 0.$$

The process is also unif. integrable, so $EM_\infty = EM_n = 1$ and

$$Var\, M_\infty = EM_\infty^2 - (EM_\infty)^2 = \frac{\sigma^2}{\mu(\mu-1)}$$

Problem 2.

Problem 2. Let $X_i, i \geq 1$ be a symmetric random walk on $\mathbb{Z}$, and let S_n be the position at time n. Consider $D_n := \max_{0 \leq k \leq n} S_k - S_n$. Now take an integer $d > 0$ and let $T := \inf\{n \geq 0 : D_n \geq d\}$.

(a) (OST) Is it true that $ES_T = 0$?

(b) Show that $ET < \infty$ for any $d > 0$.

(c) Find $E(\max_{0 \leq k \leq T} S_k)$. Does the optional stopping theorem (OST) apply to this calculation, i.e., are its assumptions satisfied?

(a)

Corollary 17.8 (Optional Stopping Theorem, Version 3). *Let (M_t) be a martingale with respect to $\{\mathcal{F}_t\}$ having bounded increments, that is $|M_{t+1} - M_t| \leq B$ for all t, where B is a non-random constant. Suppose that τ is a stopping time for $\{\mathcal{F}_t\}$ with $\mathbf{E}(\tau) < \infty$. Then $\mathbf{E}(M_\tau) = \mathbf{E}(M_0)$.*

Levin & Pers, Markov chains and mixing time, second edition.

we have $|S_n - S_{n-1}| \leq 1$, $E(\tau) < \infty$ will be proved in part (b).

Therefore, OST holds and $ES_T = ES_0 = 0$.

(c) By the definition of T we have

$$E(\max_{0 \leq k \leq T} S_k) = E(S_T + d) = d$$

$$P(T=n) = 2^{-n} \cdot \boxed{\#\text{ Paths } (t, S_t) \text{ s.t. } B_n - D_n = d,\ B_k - D_n < d \text{ for all } k < n.}$$

call this A_n

Show $\sum n \dfrac{A_n}{2^n} < \infty$?

Problem 3

Problem 3. Given a Brownian motion process $B(t), t \geq 0$ and a time value $s > 0$. For each of the following processes

(1) $X(t) = -B(t)$
(2) $X(t) = B(s-t) - B(s)$
(3) $X(t) = aB(t/a^2)$, where $a \neq 0$ is a number
(4) $X(t) = tB(1/t), t > 0$ and $X(0) = 0$

show that $X(t)$ is a Gaussian process with $\mathrm{Cov}(X(t_1)X(t_2)) = t_1 \wedge t_2$. Argue further that $X(t)$ is a Brownian motion.

Solution:

(a) Clearly $X(t)$ is a Gaussian process: for any $t_1 < \dots < t_n < T$

$$(X(t_1), \dots, X(t_n)) = -(B(t_1), \dots, B(t_n))$$

is a Gaussian vector. Next

$$\mathrm{Cov}(X(s), X(t)) = \mathrm{Cov}(-B(s), -B(t)) = \mathrm{Cov}(B(s), B(t)) = s \wedge t$$

(b) $X(t) = B(T-t) - B(T)$, T finite.

For $t_1 < \dots < t_n \leq T$

$$(X(t_1), \dots, X(t_n)) = (B(T-t_1) - B(T), \dots, B(T-t_n) - B(T))$$

is a linear transformation of a Gaussian vector $B(T-t_1), \dots, B(T-t_n)$, $B(T)$, and so is Gaussian itself. Next,

$$\mathrm{Cov}(B(T-s) - B(T), B(T-t) - B(T)) = (T-s)\wedge(T-t) - (T-s)\wedge T - T\wedge(T-t) - \underset{\mathrm{Var}(B(T))}{T}$$

$$= (T-s)\wedge(T-t) + s + t + T = t \wedge s$$

(c) Take $t_1 < \dots < t_n \leq T$; the vector $(X(t_1), \dots, X(t_n)) = c\,(B(t_1/c^2), \dots, B(t_n/c^2))$ is Gaussian. Next

$$\mathrm{Cov}(X(s), X(t)) = c^2\, \mathrm{Cov}(B(s/c^2), B(t/c^2)) = c^2\, \frac{s}{c^2} \wedge \frac{t}{c^2} = s \wedge t.$$

(d) $X(t)$ is a Gaussian process w 0 mean.

$$\mathrm{Cov}(X(s), X(t)) = st\, \mathrm{Cov}(B(\tfrac{1}{s}), B(\tfrac{1}{t})) = st\left(\tfrac{1}{s} \wedge \tfrac{1}{t}\right) = s \wedge t.$$

Problem 4

Problem 4. For a standard Brownian motion process on $\mathbb{R}_+$, define $M(t) = \max_{s \le t} B(s), t \ge 0$.

(a) Show that $M(t) \stackrel{d}{=} |B(t)|$ (the RVs $M(t)$ and $|B(t)|$ have the same distribution). (Hint: for instance, use the reflection principle)

(b) Show moreover that $M(t) - B(t) \stackrel{d}{=} M(t)$.

a) Define the stopping time

$$T_a := \inf\{t : B(t) = a\} \quad \text{for some fixed } a \in \mathbb{R}.$$

Then

$$\begin{aligned} P(M(t) \geq a) &= P(T_a \leq t) && \text{Definition of } T_a. \\ &= 2P(B(t) \geq a) && \text{Reflecting principle} \\ &= P(|B(t)| \geq a) && B(t) \sim N(0,t) \end{aligned}$$

b)

$$\begin{aligned} M(t) - B(t) &= \max_{0 \le s \le t} B(s) - B(t) \\ &= \max_{0 \le s \le t} -[B(t) - B(s)] \\ &= \max_{0 \le s \le t} -B(t-s) && B(\cdot) \text{ standard Brownian motion} \\ &\stackrel{d.}{=} \max_{0 \le s \le t} B(s) && B(t) \sim N(0,t) \stackrel{d}{=} -B(t) \end{aligned}$$

Therefore, $M(t) - B(t) \stackrel{d.}{=} M(t)$

Problem 5.

Problem 5. Let $B(t)$ be a standard BM and let $a > 0$. Define the process $X(t) = e^{-at}B(e^{2at}), t \geq 0$.

(a) Show that $X(t)$ Gaussian, i.e., that every finite-dimensional sample forms $X(t_1), \ldots, X(t_n)$ a Gaussian vector.

(b) Find $EX(t)$ and $\mathrm{Cov}(X(t), X(s))$.

a) Take $0 < t_1 < \cdots < t_n$, we have

$$\begin{bmatrix} e^{-at_1} B(e^{2at_1}) \\ \vdots \\ e^{-at_n} B(e^{2at_n}) \end{bmatrix} = \begin{bmatrix} e^{-at_1} & & & \\ & e^{-at_2} & & \\ & & \ddots & \\ & & & e^{-at_n} \end{bmatrix} \cdot \begin{bmatrix} B(e^{2at_1}) \\ \vdots \\ B(e^{2at_2}) \end{bmatrix}$$

is a finite linear combination of a Gaussian vector $\begin{bmatrix} B(e^{2at_1}) \\ \vdots \\ B(e^{2at_2}) \end{bmatrix}$.

b) $E(X(t))$

$= E(e^{-at} B(e^{2at})) = e^{-at} E(B(e^{2at})) = 0$

$\mathrm{Cov}(X(t), X(s))$

$= \mathrm{Cov}[e^{-at} B(e^{2at}), e^{-as} B(e^{2as})]$

$= e^{-at-as} \mathrm{Cov}[B(e^{2at}), B(e^{2as})]$

$= e^{-a(t+s)} \cdot (e^{2at} \wedge e^{2as}) = e^{-a(t+s) + 2a(t\wedge s)}$

$= e^{-a|s-t|}$