**ENEE620-24. Home assignment 5. Date due December 8, 11:59pm EDT.**

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Please submit your work as a **single PDF file** to ELMS (under the "Assignments" tab)
- Papers submitted as multiple pictures of individual pages are difficult for grading and **will not be accepted**.
- Justification of solutions is required.
- Each problem is worth 10 points.

**Problem 1.** Consider a standard Galton-Watson branching process $(X_n)_{n\ge 0}$ with $X_0 = 1$ and $X_{n+1} = \sum_{k=1}^{X_n} Z_{n+1}^{(k)}$, where $(Z_n^{(k)}, n \ge 1, k \ge 1)$ is a collection of iid RVs with finite expectation $\mu$ and variance $\sigma^2$, taking values in $\mathbb{N}_0$.

(a) Prove that $M_n := X_n/\mu^n$ forms a martingale with respect to the natural filtration $(\mathcal{F}_n)_n$ defined by $\mathcal{F}_n = \sigma(X_0, X_1, \dots, X_n)$.

(b) Show that $E(X_{n+1}^2|\mathcal{F}_n) = \mu^2 X_n^2 + \sigma^2 X_n$.

(c) Show that $M_n$ is bounded in $L^2$ (i.e., $\sup_{n\ge 1} EM_n^2 < \infty$) if and only if $\mu > 1$.

(d) Show that for $\mu > 1$, $\text{Var}(M_\infty) = \sigma^2/(\mu(\mu-1))$.

**Problem 2.** Let $X_i, i \ge 1$ be a symmetric random walk on $\mathbb{Z}$, and let $S_n$ be the position at time $n$. Consider $D_n := \max_{0\le k\le n} S_k - S_n$. Now take an integer $d > 0$ and let $T := \inf\{n \ge 0 : D_n \ge d\}$.

(a) (OST) Is it true that $ES_T = 0$?

(b) Show that $ET < \infty$ for any $d > 0$.

(c) Find $E(\max_{0\le k\le T} S_k)$. Does the optional stopping theorem (OST) apply to this calculation, i.e., are its assumptions satisfied?

**Problem 3.** Given a Brownian motion process $B(t), t \ge 0$ and a time value $s > 0$. For each of the following processes

(1) $X(t) = -B(t)$
(2) $X(t) = B(s-t) - B(s)$
(3) $X(t) = aB(t/a^2)$, where $a \ne 0$ is a number
(4) $X(t) = tB(1/t), t > 0$ and $X(0) = 0$

show that $X(t)$ is a Gaussian process with $\text{Cov}(X(t_1)X(t_2)) = t_1 \wedge t_2$. Argue further that $X(t)$ is a Brownian motion.

**Problem 4.** For a standard Brownian motion process on $\mathbb{R}_+$, define $M(t) = \max_{s\le t} B(s), t \ge 0$.

(a) Show that $M(t) \overset{d}{=} |B(t)|$ (the RVs $M(t)$ and $|B(t)|$ have the same distribution). (Hint: for instance, use the reflection principle)

(b) Show moreover that $M(t) - B(t) \overset{d}{=} M(t)$.

**Problem 5.** Let $B(t)$ be a standard BM and let $a > 0$. Define the process $X(t) = e^{-at}B(e^{2at}), t \ge 0$.

(a) Show that $X(t)$ Gaussian, i.e., that every finite-dimensional sample forms $X(t_1), \dots, X(t_n)$ a Gaussian vector.

(b) Find $EX(t)$ and $\text{Cov}(X(t), X(s))$.