Problem 1.

Problem 1. Consider a Galton-Watson branching process $\{X_n\}_{n\geq 0}$, where $X_0 = 1$ and X_n equals the population size in the n-th generation. Assume that the offspring random variable Z is supported on $\{0, 2\}$ and $P(Z = 0) = p_0, P(Z = 2) = p_2 = 1 - p_0$.

(a) What is the generating function of the distribution P_Z?

(a) The generating function $G(z) = E[z^Z] = p_0 + p_2 z^2$

(b) The expected # of children $EZ = 2p_2$

Thus $P(\mathcal{E}) = 0$ if $EZ \leq 1$ or $p_2 = 1 - p_0 \leq \frac{1}{2}$, or $p_0 \geq 1/2$

Or, using the fixed-point analyzis

$$z = G(z) \Rightarrow z = \frac{1 - \sqrt{1 - 4p_0p_2}}{2p_2}$$

$$z = p_0 + p_2 z^2$$

If $p_0 = 0$, i.e. $Z = 2$ w.p. 1, $P(\mathcal{E}) = 0$

If $p_0 \geq \frac{1}{2}$, $z = 1 \Rightarrow P(\mathcal{E}) = 1$

(c) Long-term survival. Suppose that $EZ > 1$.

$P(X_n > 0) = P(\text{survival past time } n)$

We have $P(X_n = 0) = G_n(0)$

$P(X_n > 0) = 1 - P(X_n = 0) = 1 - G_n(0)$; We also know that $P(\mathcal{E}) = \lim_{n\to\infty} G_n(0)$

$\lim P(X_n > 0) = 1 - \lim_{n\to\infty} G_n(0) = 1 - q$.

Problem 2.

Problem 2. Let X, Y be independent RVs.

(a) Assume that EX exists and that $X \stackrel{d}{=} Y$, i.e., X and Y have the same distribution. Show that $E(X|X+Y) = E(Y|X+Y) = (X+Y)/2$ a.s.

(b) Assume that EX^2 and EY^2 are finite. Suppose that X is symmetric, i.e., that X and $-X$ have the same distribution, $X \stackrel{d}{=} -X$. Show that $E[(X+Y)^2|X^2+Y^2] = X^2+Y^2$ a.s.

(a) The pair (X,Y) has the same distribution as (Y,X).
This implies that $(X, X+Y) \stackrel{d}{=} (Y, X+Y)$ and $X\mathbb{1}_A \stackrel{d}{=} Y\mathbb{1}_A$ for all $A \in \sigma(X+Y)$. Therefore

$$E X\mathbb{1}_A = EY\mathbb{1}_A$$

$$E[X|X+Y] = E[Y|X+Y] = E[\tfrac{1}{2}(X+Y)|X+Y] = \frac{X+Y}{2}$$

explanation

$$E[\tfrac{X+Y}{2}|X+Y] = \tfrac{1}{2}E[X|X+Y] + \tfrac{1}{2}E[Y|X+Y]$$

(b) By independence of X, Y and $X \stackrel{d}{=} -X$, we conclude that

$$(X,Y) \stackrel{d}{=} (-X,Y)$$

Then $(XY, X^2+Y^2) \stackrel{d}{=} (-XY, X^2+Y^2)$

$$E[XY|X^2+Y^2] = E[-XY|X^2+Y^2] = 0 \text{ a.s.}$$

and $E[(X+Y)^2|X^2+Y^2] = E[X^2+2XY+Y^2|X^2+Y^2] = X^2+Y^2$ a.s.

Problem 3.

Problem 3. Let $X_n, n \geq 1$ be i.i.d. RVs with $X \sim \mathrm{Exp}(\lambda)$ (exponential distribution). Form the partial sums $S_n = X_1 + \cdots + X_n, n \geq 1$ and put $S_0 = 0$. Consider the sequence of RVs $Z_n = \sqrt{S_n} - \sqrt{S_{n-1}}, n = 1, 2, \ldots$. Does this sequence converge in probability, almost surely, or in L_1? If yes, identify the limit.

By SLLN, $\frac{S_n}{n} \underset{n\to\infty}{\rightarrow} EX = \frac{1}{\lambda}$

By Newton's binomial $(x+a)^{1/2} = \sqrt{a} + \frac{x}{2\sqrt{a}} - o(x)$

$\sqrt{S_{n+1}} = \sqrt{S_n + X_{n+1}} = \sqrt{S_n} + \frac{X_{n+1}}{2\sqrt{S_n}}$, so we can approximate

$Z_n = \frac{X_{n+1}}{2\sqrt{S_n}} = \frac{X_n}{2\sqrt{n}\sqrt{\frac{S_n}{n}}} = \frac{X_n}{2\sqrt{\frac{n}{\lambda}}} \pm \delta$ a.s. starting with some n.

Moreover

$p_n = P(Z_n > \varepsilon) = P\left(X_{n+1} > 2\varepsilon\sqrt{\frac{n}{\lambda}}\right) = \exp\left(-\lambda \cdot 2\varepsilon\sqrt{\frac{n}{\lambda}}\right)$, so

$\sum p_n < \infty$, and by Borel-Cantelli, $P(Z_n > \varepsilon \text{ i.o.}) = 0$.

This implies that $Z_n \to 0$ a.s., so also $Z_n \xrightarrow{P} 0$.

Writing $EZ_n = \frac{\lambda\sqrt{\lambda}}{2\sqrt{n}} \pm \delta$, we also argue that $EZ_n \to 0$, so

$Z_n \to 0$ in L_1

Problem 4.

Problem 4. Consider a random walk on $\{0, 1, \ldots, n\}$ with transition probabilities given by

$$p_{ij} = \begin{cases} b_i, & j = i - 1 \\ a_i & j = i + 1 \\ 1 - (a_i + b_i) & j = i \\ 0 & |j - i| > 1, \end{cases}$$

where $a_0 = b_0 = a_n = b_n = 0$ and $a_i > 0, b_i > 0, i = 1, \ldots, n-1$. Suppose the walk starts in state k. What is the expected time of absorption at 0?

With no extra assumptions the expected time $= \infty$, so

Let us condition our calculation on absorption at 0 = no absorption at n

$$\Rightarrow \quad p_{n-1,n-2} = \frac{b_{n-1}}{1-a_{n-1}}; \quad p_{n-1,n-1} = \frac{1-a_{n-1}-b_{n-1}}{1-a_{n-1}}$$

Let $T_k = \mathbb{E}[\text{time of absorption at } 0 \text{ starting in state } k]$

We have $T_0 = 0$,

$$T_1 = 1 + a_1 T_2 + b_1 \overset{0}{\overset{\shortparallel}{T_0}} + (1 - a_1 - b_1) T_1$$

$$T_2 = 1 + a_2 T_3 + b_2 T_1 + (1 - a_2 - b_2) T_2$$

$\ldots$

$$T_{n-2} = 1 + a_{n-2} T_{n-1} + b_{n-2} T_{n-3} + (1 - a_{n-2} - b_{n-2}) T_{n-2}$$

$$T_{n-1} = 1 + \frac{b_{n-1}}{1-a_{n-1}} T_{n-2} + \frac{1-a_{n-1}-b_{n-1}}{1-a_{n-1}} T_{n-1}$$

Rewriting:

$$T_1 (a_1 + b_1) = 1 + a_1 T_2$$

The values of T_k are obtained by solving this sysmem

$$T_2 (a_2 + b_2) = 1 + a_2 T_3 + b_2 T_1$$

$\ldots$

$$T_k (a_k + b_k) = 1 + a_k T_{k+1} + b_k T_{k-1}$$

$$T_{n-2}(a_{n-1} + b_{n-1}) = 1 + a_{n-2} T_{n-1} + b_{n-1} T_{n-2}$$

$$T_{n-1} b_{n-1} = 1 - a_{n-1} + b_{n-1} T_{n-2}$$

Problem 5

Problem 5. (a) Let $(X_n)_n$ be a sequence of independent RVs with $E|X_n| < \infty$ and $EX_n = 0, n \geq 1$. Show that for every fixed $k \geq 1$, the sequence

$$Z_n^{(k)} = \sum_{1 \leq i_1 < \cdots < i_k \leq n} X_{i_1} \ldots X_{i_k}, \quad n = k, k+1, \ldots$$

forms a martingale.

(b) Let $(X_n)_n$ be a sequence of integrable RVs such that

$$E(X_{n+1}|X_1, \ldots, X_n) = \frac{X_1 + \cdots + X_n}{n}, \quad n \geq 1.$$

Show that the sequence of RVs $Z_n := \frac{1}{n}(X_1 + \cdots + X_n), n \geq 1$ forms a martingale.

(a) By independence,

$$E|Z_n^{(k)}| = \left|\sum_{i_1,\ldots,i_k} \prod_{j=1}^{k} EX_{i_j}\right| \leqslant \sum \prod |EX_{i_j}| < \infty$$

Let $\mathcal{F}_n = \sigma(X_1, \ldots, X_n)$, $n \geqslant k$ (k fixed, n changes)

$$E\left[Z_{n+1}^{(k)} \mid \mathcal{F}_n\right] = E\left[\sum_{1 \leqslant i_1,\ldots,i_k \leqslant n+1} \prod_{j=1}^{k} X_{i_j} \mid \mathcal{F}_n\right]$$

$$= \sum_{i_1,\ldots,i_k} E\left[\prod_j X_{i_j} \mid \mathcal{F}_n\right] = \sum_{1 \leqslant i_1 < \cdots < i_k \leqslant n} E\left[\prod_j X_{i_j} \mid \mathcal{F}_n\right]$$

$$+ \sum_{\substack{1 \leqslant i_1 < \cdots < i_{k-1} \\ i_k = n+1}} E\left[X_{i_1} X_{i_2} \ldots X_{i_{k-1}} X_{n+1} \mid \mathcal{F}_n\right]$$

$$= Z_n^{(k)} + \sum_{\substack{1 \leqslant i_1 < \cdots < i_{k-1} \\ i_k = n+1}} X_{i_1} \ldots X_{i_{k+1}} \overbrace{E\left[X_{n+1} \mid \mathcal{F}_n\right]}^{=0} = Z_n^{(k)} \quad //$$

(b) By assumption, $E|Z_n| \leqslant \frac{1}{n} \sum_{i=1}^{n} |X_i| < \infty$

Let $\mathcal{F}_n = \sigma(Z_1, \ldots, Z_n) = \sigma(X_1, \ldots, X_n)$

$$E\left[Z_{n+1} \mid \mathcal{F}_n\right] = E\left[\frac{X_1 + \cdots + X_{n+1}}{n+1} \mid \mathcal{F}_n\right] = \frac{X_1 + \cdots + X_n}{n+1} + \frac{1}{n+1} E\left[X_{n+1} \mid \mathcal{F}_n\right]$$

$$= \frac{X_1 + \cdots + X_n}{n+1} + \frac{1}{n+1} \frac{X_1 + \cdots + X_n}{n} = \frac{X_1 + \cdots + X_n}{n} = Z_n \quad //$$