ENEE620-24. Home assignment 4. Date due November 9, 11:59pm EDT.

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Please submit your work as a **single PDF file** to ELMS (under the "Assignments" tab)

- Papers submitted as multiple pictures of individual pages are difficult for grading and **will not be accepted**.
- Justification of solutions is required.
- Each problem is worth 10 points.

Problem 1. Consider a Galton-Watson branching process $\{X_n\}_{n\ge 0}$, where $X_0 = 1$ and X_n equals the population size in the n-th generation. Assume that the offspring random variable Z is supported on $\{0, 2\}$ and $P(Z = 0) = p_0, P(Z = 2) = p_2 = 1 - p_0$.

(a) What is the generating function of the distribution P_Z?

(b) Find the extinction probability $P(E)$ in terms of p_0. In particular, what is $P(E)$ if $p_0 = 0$, $p_0 = 0.5$, and $p_0 = 1$?

(c) (Long-Term Behavior): Let $EZ > 1$ and let q be the extinction probability found in part (a). Show that the probability $\Pr(X_n > 0)$ that the population survives to generation n converges to $1 - q$ as $n \to \infty$.

Problem 2. Let X, Y be independent RVs.

(a) Assume that EX exists and that $X \overset{d}{=} Y$, i.e., X and Y have the same distribution. Show that $E(X|X + Y) = E(Y|X + Y) = (X + Y)/2$ a.s.

(b) Assume that EX^2 and EY^2 are finite. Suppose that X is symmetric, i.e., that X and $-X$ have the same distribution, $X \overset{d}{=} -X$. Show that $E[(X + Y)^2|X^2 + Y^2] = X^2 + Y^2$ a.s.

Problem 3. Let $X_n, n \ge 1$ be i.i.d. RVs with $X \sim \text{Exp}(\lambda)$ (exponential distribution). Form the partial sums $S_n = X_1+\cdots+X_n, n \ge 1$ and put $S_0 = 0$. Consider the sequence of RVs $Z_n = \sqrt{S_n}-\sqrt{S_{n-1}}, n = 1, 2, \ldots$. Does this sequence converge in probability, almost surely, or in L_1? If yes, identify the limit.

Problem 4. Consider a random walk on $\{0, 1, \ldots, n\}$ with transition probabilities given by

$$p_{ij} = \begin{cases} b_i, & j = i - 1 \\ a_i & j = i + 1 \\ 1 - (a_i + b_i) & j = i \\ 0 & |j - i| > 1, \end{cases}$$

where $a_0 = b_0 = a_n = b_n = 0$ and $a_i > 0, b_i > 0, i = 1, \ldots, n - 1$. Suppose the walk starts in state k. What is the expected time of absorption at 0?

Problem 5. (a) Let $(X_n)_n$ be a sequence of independent RVs with $E|X_n| < \infty$ and $EX_n = 0, n \ge 1$. Show that for every fixed $k \ge 1$, the sequence

$$Z_n^{(k)} = \sum_{1\le i_1<\cdots<i_k\le n} X_{i_1} \ldots X_{i_k}, \quad n = k, k + 1, \ldots$$

forms a martingale.

(b) Let $(X_n)_n$ be a sequence of integrable RVs such that

$$E(X_{n+1}|X_1, \ldots, X_n) = \frac{X_1 + \cdots + X_n}{n}, \quad n \ge 1.$$

Show that the sequence of RVs $Z_n := \frac{1}{n}(X_1 + \cdots + X_n), n \ge 1$ forms a martingale.