HW3 Solutions

Problem 1. (a) Suppose that $(X_n)_n$ is a sequence of independent RVs such that

$$P(X_n = 2^n) = P(X_n = -2^n) = 1/2, n \geq 1.$$

Does this sequence satisfy the (strong or weak) law of large numbers?

(b) Suppose that $(X_n)_n$ is a sequence of independent RVs such that

$$P(X_n = \pm n) = \frac{1}{2n \ln n},\ P(X_n = 0) = 1 - \frac{1}{n \ln n}, n \geq 2.$$

Let $S_n = X_2 + \cdots + X_n$. Is it true that S_n/n converges in probability? Is it true that S_n/n converges a.s.? If the answer is yes, identify the limit.

Problem 1:

1. Suppose that $X_{n+1} = 2^{n+1}$, $X_{n+2} = 2^{n+2}$, then irrespective of $X_1, \dots, X_n$, $S_n := \sum_{i=1}^{n+2} X_n \geq 2^{n+2}$. Since

$$P(S_n \geq 2^{n+2}) \geq \frac{1}{4}$$

it is not true that $P\left(\frac{S_n}{n} \geq \varepsilon\right) \xrightarrow[n\to\infty]{} 0$

Thus WLLN fails, and all the more, SLLN fails for $(X_n)_n$

2. $$\mathrm{Var}(S_n) = \sum_{i=2}^{n} \mathrm{Var}(X_i) = \sum_{i=2} EX_i^2 = \sum_i i^2 \frac{1}{i \ln i} = \sum_{i=2}^{n} \frac{i}{\ln i} = \frac{cn^2}{\ln n}$$

for some constant c

Using Chebyshev's inequality

$$P\left(\frac{|S_n|}{n} \geq \varepsilon\right) \leq \frac{cn^2}{n^2 \varepsilon^2 \ln n} \to 0 \quad n \to \infty$$

proving WLLN

Next, consider

$$\sum_{n=2}^{N} P(|X_n| \geq n) = \sum_n \frac{1}{n \ln n} \geq \int_2^{n+1} \frac{dx}{x \ln x} = \ln \ln (n+1) - \ln \ln 2$$

The RVs $(X_n)_n$ are independent, so the events $\{X_n \geq n\}$ are independent, and thus

$$P(|X_n| \geq n \text{ i.o.}) = 1$$

Finally, since $X_n = S_n - S_{n-1}$

$$P\left(\lim_{n\to\infty} \frac{S_n}{n} = 0\right) \leq P\left(\lim_{n\to\infty} \frac{X_n}{n} = 0\right) = 0,$$ so SLLN fails.

Problem 2. Consider a sequence of RVs $(X_n)_n$.

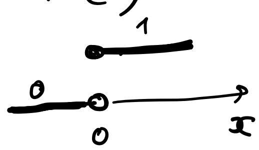
(a) If $F_{X_n}(x) = \frac{\exp(nx)}{1+\exp(nx)}, -\infty < x < \infty$, does the sequence converge in distribution? If yes, identify the limit,

(b) Same questions as part (a), with $F_{X_n}(x) = x - \frac{\sin(2\pi n x)}{2\pi n}$ for $0 \le x \le 1$, and $P(X_n < 0) = P(X_n > 1) = 0$.

Problem 2

(a) Consider $\lim_{n\to\infty} \frac{e^{nx}}{1+e^{nx}} = \begin{cases} 1 & x>0 \\ \frac{1}{2} & x=0 \\ 0 & x<0 \end{cases}$

Remember that $F_{X_n}(x) \to F(x)$ at the points of continuity,

We suspect that $F_{X_n}(x) \to$ 

For that we need to show that $P(|X_n| < \varepsilon) \to 1$ as $n \to \infty$.

Indeed,

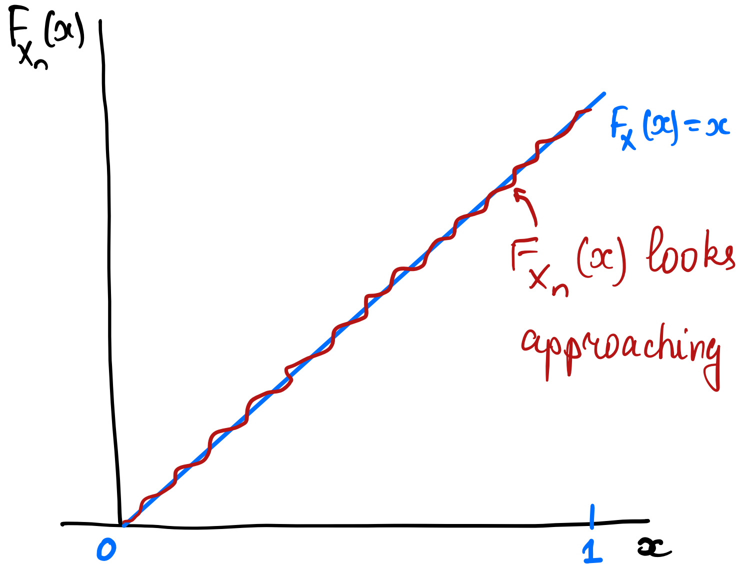
$$P(|X_n| < \varepsilon) = \frac{\exp(n\varepsilon)}{1+\exp(n\varepsilon)} - \frac{\exp(-n\varepsilon)}{1+\exp(-n\varepsilon)} \longrightarrow 1$$

so $X_n \xrightarrow{d} 0$

(b) $\sin(2\pi n x)$ oscillates, so is this even a valid (monotone nondecreasing) CDF? Yes since $F'(x) = 1 - \cos(2\pi n x) \geq 0$ for all x.

Next, $\frac{\sin 2\pi n x}{2\pi n} \to 0$ for $0 \le x \le 1$, so $F_{X_n}(x) \to x$

i.e., $X_n \xrightarrow{d} \text{Unif}[0,1]$



$F_{X_n}(x)$ looks like this, approaching the bisector as $n \to$

Problem 3. Let $(X_n)_n$ be a sequence of iid, nonnegative random variables with a common continuous distribution. Let $R_1 = 1$, $R_m = \inf\{n \geq R_{m-1} : X_n \geq \max(X_1, \ldots, X_{n-1})\}$, $m \geq 2$. Show that the sequence $(R_k)_{k\geq 1}$ forms a Markov chain. Find the transition probability matrix of this Markov chain.

Problem 3. The sequence $(R_m)_m$ forms a Markov chain.

Indeed, by def., for any realization $R_1=1, R_2=n_2, \ldots, R_k=n_k$

we have $n_1 = 1 < n_2 < \ldots < n_k$

$$X_1 \leq X_{n_2} \leq \ldots \leq X_{n_k}$$

Next, $\quad R_1=n_1, R_2=n_2, \ldots R_{k-1}=n_{k-1}$

$$P\left(R_k = n_{k-1} + m \,/\, R_1^{k-1} = n_1^{k-1}\right)$$

$$= \frac{P\left(R_1^{k-1} = n_1^{k-1},\ X_{n_{k-1}+m} \geq X_{n_{k-1}};\ X_{n_{k-1}+m-1} < X_{n_{k-1}}, \ldots, X_{n_{k-1}+1} < X_{n_{k-1}}\right)}{P\left(R_1^{k-1} = n_1^{k-1}\right)}$$

$$= P\left(R_k = n_{k-1} + m \,/\, R_{k-1} = n_{k-1}\right)$$

proving the Markov condition.

Now let us compute the transition probabilities

If $\{R_k = n_{k-1} + m \,|\, R_{k-1} = n_{k-1}\}$, then

$$X_{n_{k-1}+m} \geq X_{n_{k-1}} \quad \text{and} \quad X_{n_{k-1}+1}^{n_{k-1}+m-1} < X_{n_{k-1}}$$

Now let $F(x)$ be the common CDF of the RVs.

For any value $X_{n_{k-1}} = x$,

$$P\left(X_{n_{k-1}+1}^{n_{k-1}+m-1} < x\right) = F(x)^{m-1} \text{ by independence}$$

$$P\left(X_{n_{k-1}+m} \geq x\right) = 1 - F(x)$$

Thus

$$P\left(R_k = n_{k-1} + m \mid R_{k-1} = n_{k-1}\right) = \int_{-\infty}^{\infty} F(x)^{m-1}\left(1 - F(x)\right) dF(x)$$

$$= \frac{1}{m} - \frac{1}{m+1} = \frac{1}{m(m+1)}, \quad m = 1, 2, \ldots$$

and it does not depend on k.

Finally, $P(R_k < k) = 0$, and so

$$P = \begin{bmatrix} 0 & \frac{1}{2} & \frac{1}{2\cdot3} & \frac{1}{3\cdot4} & \cdots & \\ 0 & 0 & \frac{1}{2} & \frac{1}{2\cdot3} & \frac{1}{3\cdot4} & \cdots \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2\cdot3} & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots \end{bmatrix}$$

Problem 4. Transition probabilities of a Markov chain with 3 states satisfy

$$p_{ij} = \begin{cases} p_{1.i-j+1}, & i \geq j, \\ p_{1,j-i+1}, & j > i. \end{cases}$$

Find the matrix of transition probabilities in n steps and its limit for $n \to \infty$.

Problem 4. Let $p_{11} = a$, then the conditions of the problem imply that $p_{12} = p_{13}$, and thus they are $\frac{1-a}{2}$.

The same for every row, so

$$P = \begin{bmatrix} a & \frac{1-a}{2} & \frac{1-a}{2} \\ \frac{1-a}{2} & a & \frac{1-a}{2} \\ \frac{1-a}{2} & \frac{1-a}{2} & a \end{bmatrix}$$

If we denote $(P^n)_{1,1} = a^{(n)}$, then similarly

$$P^n = \begin{bmatrix} a^{(n)} & \frac{1-a^{(n)}}{2} & \frac{1-a^{(n)}}{2} \\ \frac{1-a^{(n)}}{2} & a^{(n)} & \frac{1-a^{(n)}}{2} \\ \frac{1-a^{(n)}}{2} & \frac{1-a^{(n)}}{2} & a^{(n)} \end{bmatrix}$$

Now use the Kolmogorov - Chapman equations:

$$a^{(n)} = p_{11}^{(n)} = \sum_{j=1}^{3} p_{1j}^{(n-1)} p_{j1} = a \cdot a^{(n-1)} + \frac{1-a^{(n-1)}}{2} \cdot (1-a) = \frac{1}{2}\Big(2a \cdot a^{(n-1)} + 1 - a - a^{(n-1)} +$$

$$+ a\, a^{(n-1)}\Big) = \frac{3a\, a^{(n-1)} - a - a^{(n-1)} + 1}{2} = \frac{3a-1}{2} a^{(n-1)} - \frac{a-1}{2}$$

Continue recursively $\rightsquigarrow$

$$= \frac{3a-1}{2}\left(\frac{3a-1}{2} a^{(n-2)} + \frac{1-a}{2}\right) + \frac{1-a}{2}$$

$$= \dots = \left(\frac{3a-1}{2}\right)^n \cdot \frac{2}{3} + \frac{1}{3} \to \begin{cases} \frac{1}{3}, & a \neq 1 \\ 1, & a = 1 \end{cases}$$

Problem 5. (a) Suppose that n points $a_1, \ldots, a_n$ are placed on a circle in the plane and numbered consecutively. For instance, think of an inscribed regular n-gon. A random walk on this point set proceeds by moving either clockwise or counterclockwise from a point to its nearest neighbor. Your task is to determine whether this walk forms a Markov chain if:

(a) it always moves deterministically clockwise;

(b) at the start it chooses the direction between clockwise and counterclockwise by coin tossing, and moves deterministically in that direction all the time;

(c) for all $i \neq 1$, it moves randomly according to $p_{i,i+1} = p, p_{i,i-1} = 1 - p$. If it lands in a_1, it returns to the vertex from which it transitioned to a_1 in the previous step.

Problem 5.

(a) $P(X_m / \underbrace{X_0^{m-1}}_{= X_0, X_1, \ldots, X_{m-1}}) = P(X_m / X_{m-1})$ because X_m is a deterministic function of X_{m-1}. Ans. : Yes

(b) The initial choice is never erased because X_m is either X_{m-1} or X_{m+1} (mod n) depending on it. Ans.: No

(c) If $X_n = 1$, $\underbrace{P(X_{n+1} / X_n, X_{n-1})}_{= \delta_{X_{n+1}, X_{n-1}}} \neq P(X_{n+1} / X_n)$

Ans. : No

Problem 6. (a) Given a sequence $(X_n)_{n\geq 0}$ of independent random variables, determine whether the following sequence forms a Markov chain: $X_0+X_1, X_1+X_2, X_2+X_3, \ldots$.

(b) A sequence $(X_n)_{n\geq 0}$ of random variables forms a Markov chain. Determine whether the following sequence forms a Markov chain: $X_0+X_1, X_2+X_3, X_4+X_5, \ldots$.

(a) In general, no. To give an example, let $P(X=1)=p=1-P(X=0)$.

$$P(X_2+X_3=1 \mid X_1+X_2=1, X_0+X_1=0) = \frac{P(X_0=X_1=0, X_2=1, X_3=0)}{P(X_0=X_1=0, X_2=1)}$$

$$= \frac{(1-p)^3 p}{(1-p)^2 p} = 1-p$$

$$P(X_2+X_3=1 \mid X_1+X_2=1) =$$

$$= \frac{P(X_3=0 \mid X_2=1, X_1=0)\,P(X_1=0, X_2=1) + P(X_3=1 \mid X_2=0, X_1=1)\,P(X_1=1, X_2=0)}{P(X_1=0, X_2=1) + P(X_1=1, X_2=0)}$$

$$= \frac{(1-p)^2 \cdot p + p^2(1-p)}{2p(1-p)} = \frac{1}{2}$$

(b) No. Argue by example, a 2-state chain should suffice

For instance, take $X = (\lambda, P)$ given by

$$P = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{4} & \frac{3}{4} \end{bmatrix}, \quad \lambda = (1/3, 2/3)$$

With these choices, $P(X_n=1) = \frac{1}{3}$, $P(X_n=2) = \frac{2}{3}$ for all $n \geq 0$

Then let $Y_m = X_m + X_{m-1}$ and compute

$P(Y_5 = 3 \mid Y_3 = 3, Y_1 = 3)$ and $P(Y_5 = 3 \mid Y_3 = 3)$