ENEE620-24. Home assignment 3. Date due October 27, 11:59pm EDT.

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Please submit your work as a **single PDF file** to ELMS (under the "Assignments" tab)
• Papers submitted as multiple pictures of individual pages are difficult for grading and **will not be accepted**.
• Justification of solutions is required.
• Each problem is worth 10 points.

Problem 1. (a) Suppose that $(X_n)_n$ is a sequence of independent RVs such that

$$P(X_n = 2^n) = P(X_n = -2^n) = 1/2, n \ge 1.$$

Does this sequence satisfy the (strong or weak) law of large numbers?

(b) Suppose that $(X_n)_n$ is a sequence of independent RVs such that

$$P(X_n = \pm n) = \frac{1}{2n \ln n},\ P(X_n = 0) = 1 - \frac{1}{n \ln n}, n \ge 2.$$

Let $S_n = X_2 + \cdots + X_n$. Is it true that S_n/n converges in probability? Is it true that S_n/n converges a.s.? If the answer is yes, identify the limit.

Problem 2. Consider a sequence of RVs $(X_n)_n$.

(a) If $F_{X_n}(x) = \frac{\exp(nx)}{1+\exp(nx)}, -\infty < x < \infty$, does the sequence converge in distribution? If yes, identify the limit,

(b) Same questions as part (a), with $F_{X_n}(x) = x - \frac{\sin(2\pi n x)}{2\pi n}$ for $0 \le x \le 1$, and $P(X_n < 0) = P(X_n > 1) = 0$.

Problem 3. Let $(X_n)_n$ be a sequence of iid, nonnegative random variables with a common continuous distribution. Let $R_1 = 1$, $R_m = \inf\{n > R_{m-1} : X_n \ge \max(X_1, \dots, X_{n-1})\}$, $m \ge 2$. Show that the sequence $(R_k)_{k \ge 1}$ forms a Markov chain. Find the transition probability matrix of this Markov chain.

Problem 4. Transition probabilities of a Markov chain with 3 states satisfy

$$p_{ij} = \begin{cases} p_{1.i-j+1}, & i \ge j, \\ p_{1,j-i+1}, & j > i. \end{cases}$$

Find the matrix of transition probabilities in n steps and its limit for $n \to \infty$.

Problem 5. (a) Suppose that n points $a_1, \dots, a_n$ are placed on a circle in the plane and numbered consecutively. For instance, think of an inscribed regular n-gon. A random walk on this point set proceeds by moving either clockwise or counterclockwise from a point to its nearest neighbor. Your task is to determine whether this walk forms a Markov chain if:

(a) it always moves deterministically clockwise;

(b) at the start it chooses the direction between clockwise and counterclockwise by coin tossing, and moves deterministically in that direction all the time;

(c) for all $i \ne 1$, it moves randomly according to $p_{i,i+1} = p, p_{i,i-1} = 1 - p$. If it lands in a_1, it returns to the vertex from which it transitioned to a_1 in the previous step.

Problem 6. (a) Given a sequence $(X_n)_{n \ge 0}$ of independent random variables, determine whether the following sequence forms a Markov chain: $X_0 + X_1, X_1 + X_2, X_2 + X_3, \dots$.

(b) A sequence $(X_n)_{n \ge 0}$ of random variables forms a Markov chain. Determine whether the following sequence forms a Markov chain:$X_0 + X_1, X_2 + X_3, X_4 + X_5, \dots$.