Problem 1

Problem 1. We say that a random variable X is constant a.s. if there is a real number a such that $P(X = a) = 1$.

For each of the following statements, determine whether it is true or false and explain why.

(1) If X is independent of itself, then X is constant a.s. (Hint: a contradiction should be easy to construct)

(2) If X is independent of X^2, then X is constant a.s. (Hint: squaring smoothes out differences)

(3) If $X, Y, X+Y$ are jointly independent, then X and Y are constants a.s. (Hint: do not think)

(4) If X and Y are independent and $X+Y$ and $X-Y$ are independent, then X and Y are constants a.s. (Hint: think Gaussian)

(1) Yes. Suppose that X is not constant, then there is $B \subset \mathcal{B}(\mathbb{R})$ such that $0 < P(X \in B) < 1$. Then

$$P(X \in B, X \in B^c) = 0 \neq P(X \in B)\, P(X \in B^c)$$

Contradiction to the assumption of independence

(2) No. Let $X = \pm 1$ with probability $\frac{1}{2}$ each. Then $P(X^2 = 1) = 1$ and $P(X = i, X^2 = 1) = P(X = i)\, P(X^2 = 1) = \frac{1}{2} \cdot 1$. Thus X and X^2 are independent, but X is not constant.

(3) Yes. If at least one of X, Y is not constant, X+Y cannot be independent of X and Y because it is a deterministic function of X and Y.

(4) No. Take $X \sim \mathcal{N}(0,1)$, $Y \sim \mathcal{N}(0,1)$ independent Then

$$X+Y \sim \mathcal{N}(0,2), \quad X-Y \sim \mathcal{N}(0,2)$$

Clearly, $EX^2 = EY^2 = \sigma^2$, so

$$\mathrm{Cov}((X+Y)(X-Y)) = E((X+Y)(X-Y)) = EX^2 - EY^2 = 0$$

Uncorrelated Gaussian RV's are independent, so $(X+Y) \perp (X-Y)$

Problem 2

Problem 2. (a) Let X and Y be RVs and suppose that $X = Y$ a.s. Suppose that EX exists. Using the definition of the expectation (integral) as a limit, prove formally that EY also exists and $EX = EY$.

(b) For any finite number of independent integrable RVs $X_1, \ldots, X_n$, $E(\prod_{i=1}^n X_i) = \prod_{i=1}^n EX_i$. Now suppose that we are given a sequence of independent integrable RVs $X_1, X_2, \ldots$. Is it always true that $E(\prod_{i=1}^\infty X_i) = \prod_{i=1}^\infty EX_i$? If not, is it sometimes true?

1. Since $X=Y$ a.s., $P(X_-=Y_-) = P(X_+=Y_+)=1$, so we can limit ourselves to $X,Y \geq 0$. We have

$$\sum_{k=1}^{n^2} \frac{k-1}{n} P\left(\frac{k-1}{n} \leq X < \frac{k}{n}\right) \uparrow EX$$

Since $X=Y$ a.s, for all n, k,

$$P\left(\frac{k-1}{n} \leq X < \frac{k}{n}\right) = P\left(\frac{k-1}{n} \leq Y < \frac{k}{n}\right)$$

and thus

$$\sum_{k=1}^{n^2} \frac{k-1}{n} P\left(\frac{k-1}{n} \leq X < \frac{k}{n}\right) = \sum_{k=1}^{n^2} \frac{k-1}{n} P\left(\frac{k-1}{n} \leq Y < \frac{k}{n}\right)$$

The limit on $n \to \infty$ on the left exists

$\Rightarrow$ the limit on the right also exists

By def., this limit is EY, which therefore exists and equals EX

Problem 3.

Problem 3. Let X be an RV with $EX^2 < \infty$. Show that

$$P(X - EX \geq \epsilon) \leq \frac{\text{Var}(X)}{\text{Var}(X) + \epsilon^2}.$$

Give examples to show that this bound is sometimes attained.

Since $EX^2 < \infty$, also $EX < \infty$. Assume that $EX = 0$.

$\varepsilon = \varepsilon - EX = E(\varepsilon - X) \leqslant E(\varepsilon - X)\mathbb{1}_{\{X<\varepsilon\}}$ (limiting integration to the positive part)

Cauchy-Schwarz

$\leqslant \sqrt{E(X-\varepsilon)^2}\sqrt{P(X<\varepsilon)} = \sqrt{(\text{Var}\,X + \varepsilon^2)}\sqrt{1 - P(X \geqslant \varepsilon)}$

$$1 - P(X \geqslant \varepsilon) \geqslant \frac{\varepsilon^2}{\text{Var}\,X + \varepsilon^2}$$, as required

If $EX \neq 0$, consider $Y = X - EX$, so $\text{Var}(Y) = \text{Var}(X)$. As just shown,

$$P(X - EX \geqslant \varepsilon) = P(Y \geqslant \varepsilon) \leqslant \frac{\text{Var}(Y)}{\text{Var}(Y) + \varepsilon^2}$$

To construct an RV for which this bound is attained with equality, let

$$X = \begin{cases} \varepsilon & \text{with prob. } \frac{1}{1+\varepsilon^2} \\ -\frac{1}{\varepsilon} & \text{with prob } \frac{\varepsilon^2}{1+\varepsilon^2} \end{cases}, \quad \text{so } EX = 0$$

$$\text{Var}(X) = EX^2 = \frac{\varepsilon^2}{1+\varepsilon^2} + \frac{\varepsilon^2}{1+\varepsilon^2}\frac{1}{\varepsilon^2} = 1$$

$$P(X \geqslant \varepsilon) = \frac{1}{1+\varepsilon^2} = \frac{\text{Var}(X)}{\text{Var}(X) + \varepsilon^2}$$

Problem 4

Problem 4. (a) Let X, Y be RVs. Show that if $E\max(X,Y)$ and $E\min(X,Y)$ exist, then also EX and EY exist, and moreover

$$EX + EY = E\max(X,Y) + E\min(X,Y).$$

(b) Furthermore, let $X_1, \dots, X_n$ be RVs with finite expectations. Show that

$$E\max\{X_1, \dots, X_n\} \geq \max\{EX_1, \dots, EX_n\}$$

and

$$E\min\{X_1, \dots, X_n\} \leq \min\{EX_1, \dots, EX_n\}$$

(c) Let $(X_n)_n$ be a sequence of nonnegative RVs. We are given that $P(\sum_{n=1}^{\infty} X_i < \infty) = 1$. Show that in this case also

$$E\Big(\sum_{n=1}^{\infty} X_n\Big) = \sum_{n=1}^{\infty} EX_n.$$

$\{\max(X,Y) \leq x\} = \{\omega : Y(\omega) \leq X(\omega) \leq x\} \cup \{\omega : X(\omega) \leq Y(\omega) \leq x\} \in \mathcal{F}$

so $\max(X,Y)$ is an RV and so is $\min(X,Y)$

Next we have

$$X + Y = \max(X,Y) + \min(X,Y) \quad a.s.$$

Taking expectations on Both sides, we obtain the claim

(b) To prove the first inequality, we observe that for all i

$$X_i \leq \max\{X_1, \dots, X_n\}$$

Taking expectations on Both sides,

$$EX_i \leq E\max\{X_1, \dots, X_n\} \quad \text{for all } i = 1, \dots, n$$

$$\therefore \max\{EX_1, \dots, EX_n\} \leq E\max\{X_1, \dots, X_n\}$$

The second inequality is analogous

(c) This follows by the monotone convergence theorem applied to $E \lim_{N\to\infty} \sum_{n=1}^{N} X_n$. Since $P(\lim_{N\to\infty} \sum_{n=1}^{N} X_n < \infty) = 1$

Problem 5

Problem 5. Let $F(x)$ be a distribution function (CDF).
(a) Find $\int_{\mathbb{R}} F(x)dF(x)$ (Hint: rather than thinking, change the variable)
(b) Find $\int_{\mathbb{R}} F^k(x)dF^n(x)$, where n and k are some natural numbers.

(c) Show that any distribution function satisfies the following relations:

$$(i) \lim_{x\to+\infty} x\int_x^{\infty}\frac{dF(y)}{y} = 0; \quad (ii) \lim_{x\to-\infty} x\int_{-\infty}^{x}\frac{dF(y)}{y} = 0$$
$$(iii) \lim_{x\uparrow 0} x\int_{-\infty}^{x}\frac{dF(y)}{y} = 0; \quad (iv) \lim_{x\downarrow 0} x\int_{x}^{\infty}\frac{dF(y)}{y} = 0$$

(a) With $y = F(x)$ the integral becomes $\int_0^1 y\,dy = \frac{1}{2}$

(b) With $y = F(x)$, $\int_{-\infty}^{\infty} F^k(x)\,dF^n(x) = \int_0^1 y^k\,dy^n = n\int_0^1 y^{k+n-1}\,dy = \frac{n}{n+k}$

(c)
(i) if $y \geq x > 0$, $\frac{x}{y} \leq 1$, so $0 \leq x\int_x^{\infty}\frac{dF(y)}{y} \leq \int_x^{\infty} dF(y) \xrightarrow[x\to\infty]{} 0$

(ii) if $y \leq x < 0$, $0 < \frac{x}{y} \leq 1$, so $0 \leq x\int_{-\infty}^{x}\frac{dF(y)}{y} \leq \int_{-\infty}^{x} dF(y) \xrightarrow[x\to-\infty]{} 0$

(iii) For $x < 0$, $y < -\sqrt{|x|}$, $\frac{-\sqrt{|x|}}{y} < 1$.

$$0 \leq x\int_{-\infty}^{x}\frac{dF(y)}{y} = x\int_{-\infty}^{-\sqrt{|x|}}\frac{dF(y)}{y} + x\int_{-\sqrt{|x|}}^{x}\frac{dF(y)}{y} \leq \int_{-\infty}^{-\sqrt{|x|}} dF(y) - \sqrt{|x|}\int_{-\sqrt{|x|}}^{x} dF(y)$$

$$= F(-\sqrt{|x|}) - \sqrt{|x|}\left(F(x) - F(-\sqrt{|x|})\right) \xrightarrow[x\uparrow 0]{} 0$$

(iv) For $x>0$

$$0 \le x \int_{x}^{\infty} \frac{dF(y)}{y} = x \int_{x}^{\sqrt{x}} \frac{dF(y)}{y} + x \int_{\sqrt{x}}^{\infty} \frac{dF(y)}{y} \le \int_{x}^{\sqrt{x}} dF(y) + \sqrt{x} \int_{\sqrt{x}}^{\infty} dF(y)$$

$$= F(\sqrt{x}) - F(x) + \sqrt{x}\,(1 - F(\sqrt{x})) \xrightarrow[x \downarrow 0]{} 0$$