**ENEE620-23. Home assignment 2. Date due September 29, 11:59pm EDT.**

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Please submit your work as a **single PDF file** to ELMS (under the "Assignments" tab)
- Papers submitted as multiple pictures of individual pages are difficult for grading and **will not be accepted**.
- Justification of solutions is required.
- Each problem is worth 10 points.

**Problem 1.** We say that a random variable $X$ is constant a.s. if there is a real number $a$ such that $P(X = a) = 1$.

For each of the following statements, determine whether it is true or false and explain why.

(1) If $X$ is independent of itself, then $X$ is constant a.s. (Hint: a contradiction should be easy to construct)

(2) If $X$ is independent of $X^2$, then $X$ is constant a.s. (Hint: squaring smoothes out differences)

(3) If $X, Y, X + Y$ are jointly independent, then $X$ and $Y$ are constants a.s. (Hint: do not think)

(4) If $X$ and $Y$ are independent and $X + Y$ and $X - Y$ are independent, then $X$ and $Y$ are constants a.s. (Hint: think Gaussian)

**Problem 2.** (a) Let $X$ and $Y$ be RVs and suppose that $X = Y$ a.s. Suppose that $EX$ exists. Using the definition of the expectation (integral) as a limit, prove formally that $EY$ also exists and $EX = EY$.

(b) For any finite number of independent integrable RVs $X_1, \dots, X_n$, $E(\prod_{i=1}^n X_i) = \prod_{i=1}^n EX_i$. Now suppose that we are given a sequence of independent integrable RVs $X_1, X_2, \dots$ . Is it always true that $E(\prod_{i=1}^\infty X_i) = \prod_{i=1}^\infty EX_i$ ? If not, is it sometimes true?

**Problem 3.** Let $X$ be an RV with $EX^2 < \infty$. Show that

$$P(X - EX \ge \epsilon) \le \frac{\text{Var}(X)}{\text{Var}(X) + \epsilon^2}.$$

Give examples to show that this bound is sometimes attained.

**Problem 4.** (a) Let $X, Y$ be RVs. Show that if $E\max(X, Y)$ and $E\min(X, Y)$ exist, then also $EX$ and $EY$ exist, and moreover

$$EX + EY = E\max(X, Y) + E\min(X, Y).$$

(b) Furthermore, let $X_1, \dots, X_n$ be RVs with finite expectations. Show that

$$E\max\{X_1, \dots, X_n\} \ge \max\{EX_1, \dots, EX_n\}$$

and

$$E\min\{X_1, \dots, X_n\} \le \min\{EX_1, \dots, EX_n\}$$

(c) Let $(X_n)_n$ be a sequence of nonnegative RVs. We are given that $P(\sum_{n=1}^\infty X_i < \infty) = 1$. Show that in this case also

$$E\Big(\sum_{n=1}^\infty X_n\Big) = \sum_{n=1}^\infty EX_n.$$

**Problem 5.** Let $F(x)$ be a distribution function (CDF).

(a) Find $\int_{\mathbb{R}} F(x)dF(x)$ (Hint: rather than thinking, change the variable)

(b) Find $\int_{\mathbb{R}} F^k(x)dF^n(x)$, where $n$ and $k$ are some natural numbers.

(c) Show that any distribution function satisfies the following relations:

$$(i) \lim_{x\to+\infty} x \int_x^{\infty} \frac{dF(y)}{y} = 0; \quad (ii) \lim_{x\to-\infty} x \int_{-\infty}^{x} \frac{dF(y)}{y} = 0$$

$$(iii) \lim_{x\uparrow 0} x \int_{-\infty}^{x} \frac{dF(y)}{y} = 0; \quad (iv) \lim_{x\downarrow 0} x \int_x^{\infty} \frac{dF(y)}{y} = 0$$