HW1 - 24 Solutions

Problem 1

Problem 1. Let $(A_n)_n, (B_n)_n$ be sequences of subsets of Ω.

(a) Show that

$$\limsup_{n\to\infty}(A_n \cup B_n) = (\limsup_{n\to\infty} A_n) \cup (\limsup_{n\to\infty} B_n).$$

(b) If $A_1 \subset A_2 \subset \ldots A_n \subset \ldots$, what is $\limsup_{n\to\infty} A_n$? What is $\liminf_{n\to\infty} A_n$?
If $A_1 \supset A_2 \supset \ldots A_n \supset \ldots$, what is $\limsup_{n\to\infty} A_n$? What is $\liminf_{n\to\infty} A_n$?

(a) $$\limsup_{n\to\infty}(A_n \cup B_n) = \bigcap_{m\geq 1}\bigcup_{n\geq m} A_n \cup B_n = \bigcap_{m\geq 1}\left(\left(\bigcup_{n\geq m} A_n\right) \cup \left(\bigcup_{n\geq m} B_n\right)\right)$$

$$= \left(\bigcap_{m\geq 1}\bigcup_{n\geq m} A_n\right) \cup \left(\bigcap_{m\geq 1}\bigcup_{n\geq m} B_n\right) = \left(\limsup_{n\to\infty} A_n\right) \cup \left(\limsup_{n\to\infty} B_n\right)$$

(b) With $A_1 \subset A_2 \subset \ldots$, $\bigcup_{n\geq m} A_n = \bigcup_{n\geq m+1} A_n$, so

$$\limsup_{n\to\infty} A_n = \bigcap_{m\geq 1}\bigcup_{n\geq m} A_n = \bigcup_{m\geq 1} A_m$$

Likewise, $\bigcap_{n\geq m} A_n = A_m$ for any $m\geq 1$, so

$$\liminf_{n\to\infty} A_n = \bigcup_{m\geq 1}\bigcap_{n\geq m} A_m = \bigcup_{m\geq 1} A_m$$

Now let $A_1 \supset A_2 \supset \ldots$, then $\bigcup_{n\geq m} A_n = A_n$, $\bigcap_{n\geq m} A_n = \bigcap_{n\geq m+1} A_n$, so

$$\limsup_{n\to\infty} A_n = \bigcap_{m\geq 1}\bigcup_{n\geq m} A_n = \bigcap_{m\geq 1} A_m$$

$$\liminf_{n\to\infty} A_n = \bigcup_{m\geq 1}\bigcap_{n\geq m} A_m = \bigcap_{m\geq 1} A_m$$

Problem 2

Problem 2. Let $\Omega = \{1,2,3,4,5,6\}$ and $A = \{2,4\}, B = \{6\}$.

(a) What is the σ-algebra generated by the pair A, B on Ω?

(b) The σ-algebra generated by an RV X on the sample space Ω is defined as

$$\sigma(X) = \sigma(\{\omega : X(\omega) \leq x\}, x \in \mathbb{R}),$$

i.e., the σ-algebra generated by all the subsets $\{\omega : X(\omega) \leq x\}, x \in \mathbb{R}$.

Let $\Omega = \{1,2,3,4,5,6\}$. Consider a random variable on Ω given by $X(1) = 0.5, X(2) = 1, X(3) = 0.5, X(4) = X(5) = X(6) = 1.5$. Give an explicit description of $\sigma(X)$, i.e., list all subsets in $\sigma(X)$.

(a) $\{\Omega, \emptyset, \{2,4\}, \{1,3,5,6\}, \{6\}, \{1,2,3,4,5\}, \{2,4,6\}, \{1,3,5\}\}$

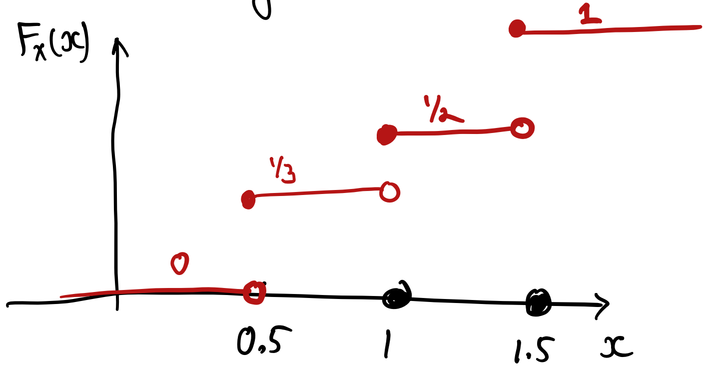
Assuming uniform P on Ω

(b) $\{\omega : X(\omega) \leq 0\} = \emptyset$

$\{X \leq 0.5\}$ $\{1,3\}$

$\{X \leq 1\}$ $\{1,2,3\}$

$\{X \leq 1.5\}$ Ω



$\sigma(X) = \{\emptyset, \Omega, \{1,3\}, \{2,4,5,6\}, \{1,2,3\}, \{4,5,6\}, \{1,3,4,5,6\}, \{2\}\}$

Problem 3

Problem 3. Consider the standard probability space on $\Omega = (0,1]$, i.e., the triple $(\Omega, \mathcal{B}(\Omega), \lambda)$, where $\mathcal{B}(\Omega)$ is the Borel σ-algebra and λ is the Lebesgue measure. Define the following random variables:

- $X_1(\omega) = 0$ for all $\omega \in \Omega$,
- $X_2(\omega) = \mathbb{1}_{\{0.2\}}(\omega)$ ($X_2(\omega) = 1$ if $\omega = 0.2$ and 0 otherwise),
- $X_3(\omega) = \mathbb{1}_{\mathbb{Q}}(\omega)$ ($X_3(\omega)$ =1 if ω is rational and 0 if it is not).

What is $\sigma(X_i)$ for $i = 1, 2, 3$, where the notation is explained in Problem 2b ?

$\sigma(X_1) = \{\emptyset, \Omega\}$ since $\{\omega : X(\omega) < 0\} = \emptyset$

$\{\omega : X(\omega) \leq 0\} = \Omega$

$\sigma(X_2)$ $\{\omega : X(\omega) < 0\} = \emptyset$

$\{\omega : X(\omega) < 0.2\} = \Omega \setminus \{0.2\}$

$\{\omega : X(\omega) \leq 0.2\} = \Omega$

So the σ-algebra $\sigma(X_2) = \{\emptyset, \Omega, \{0.2\}, \Omega \setminus \{0.2\}\}$

Consider X_3 :

$\{\omega : X_3 \leq x\} = \emptyset$ if $x < 0$

$\{\omega : X_3 \leq x\} = (\mathbb{Q} \cap (0,1])^c$ if $x < 1$

$\{\omega : X_3 \leq x\} = \mathbb{Q} \cap (0,1]$ if $x \geq 1$

$\therefore$ $\sigma(X_3) = \{\emptyset, (0,1], (\mathbb{Q} \cap (0,1])^c, \mathbb{Q} \cap (0,1]\}$

Problem 4. (a) Show that if X, Y are RVs on some probability space, then $X + Y$ is also an RV on the same space.

(b) Let $X \sim \text{Unif}[0, 2]$ (a uniform RV on $[0, 2]$). Define $Y = \max\{1, X\}$ and $Z = \min\{X, X^2\}$. Show that Y and Z are RVs and find their distribution functions.

(a) Let $(\Omega, \mathcal{F}, P)$ be the probability space in question.

We need to show that $\{\omega : X(\omega) + Y(\omega) \leq x\} \in \mathcal{F}$ for all real x

$$\{X + Y \leq x\} = \bigcup_{z \in \mathbb{Q}} \underbrace{\left(\{X \leq z\} \cap \{Y \leq x - z\}\right)}_{\in \mathcal{F}} \qquad (1)$$

By the "countable unions" property of $\mathcal{F}$, this union is a well-defined event.

If $X + Y = t$, where $t \notin \mathbb{Q}$ and $t < x$, take a rational number z s.t. $X(\omega) < z < X(\omega) + x - t$ (possible because $\mathbb{Q}$ is dense in $\mathbb{R}$)

$$Y(\omega) = t - X(\omega) < x - z$$

Thus, it suffices to consider events of the form (1) with rational z.

(b) For any pair of RVs (U, V) the functions $\max(U, V)$ and $\min(U, V)$ are also RVs. Indeed

$$\{\max(U, V) \leq z\} = \{U \leq z\} \cap \{V \leq z\}$$

$$\{\min(U, V) \leq z\} = \{U \leq z\} \cup \{V \leq z\}$$

since 1 and X^2 are RVs, this shows that so are Y and Z

To compute F_Y, F_Z, we do a calculation from the definition:

For $x<1$, $F_Y(x)=0$

For $1 \le x \le 2$, $Y \le x$ if and only if $X \le x$, which happens with probability $\frac{x}{2}$.

Thus,
$$F_Y(x) = \begin{cases} 0 & \text{if } x \le 1 \\ \frac{x}{2} & \text{if } 1 \le x \le 2 \\ 1 & \text{if } x > 2. \end{cases}$$

For Z, we note that if $x<1$, then $x^2 < x$, so

$Z \le x$ iff $X^2 \le x$.

$$P(X^2 \le x) = \frac{\sqrt{x}}{2}, \quad 0 \le x < 1$$

For $1 \le x \le 2$, $Z \le x$ iff $X \le x$.

$$P(Z \le x) = P(X \le x) = \frac{x}{2}, \quad 1 \le x \le 2$$

Overall

$$F_Z(x) = \begin{cases} 0 & x<0 \\ \sqrt{x}/2 & 0 \le x < 1 \\ x/2 & 1 \le x < 2 \\ 1 & x \geq 2. \end{cases}$$

Problem 5. (How do exponential RVs behave in the long run?) Recall that the pdf of the exponential distribution $\text{Exp}(\lambda)$ is given by $f(x) = \lambda e^{-\lambda x}\mathbb{1}_{\{x\geq 0\}}$ for some $\lambda > 0$. Consider a sequence of independent exponential RVs $X_n, n \geq 1$, each of which has pdf $f(x)$.

(a) Given an $\epsilon > 0$ find

$$P\Big(\Big\{\omega : \frac{\lambda}{\log n} X_n(\omega) > 1 - \epsilon\Big\} \text{ i.o.}\Big).$$

(b) Given an $\epsilon > 0$ find

$$P\Big(\Big\{\omega : \frac{\lambda}{\log n} X_n(\omega) > 1 + \epsilon\Big\} \text{ i.o.}\Big).$$

(c) From the previous results, what is the probability of the event $\{\omega : \limsup_n \frac{\lambda X_n}{\log n} = 1\}$?

(a) Use $F_{Exp(\lambda)}(x) = 1 - e^{-\lambda x}$ to compute

$$\sum_{n\geq 1} P\left(\frac{\lambda X_n}{\log n} > 1-\varepsilon\right) = \sum_{n\geq 1} P\left(X_n > \frac{(1-\varepsilon)}{\lambda}\log n\right) = \sum_{n\geq 1} \exp\left(-\lambda \frac{1-\varepsilon}{\lambda}\log n\right)$$

$$= \sum_{n\geq 1} n^{-(1-\varepsilon)} = \infty$$

$\therefore P\left(\frac{\lambda X_n}{\log n} > 1-\varepsilon \text{ i.o.}\right) = 1$ By independence of $(X_n)_n$.

(b) But

$$\sum_n P\left(\frac{\lambda X_n}{\log n} > 1+\varepsilon\right) = \sum_n n^{-(1+\varepsilon)} < \infty$$

$\therefore P\left(\frac{\lambda X_n}{\log n} > 1+\varepsilon \text{ i.o.}\right) = 0$

(c) $\therefore P\left(\omega : \limsup \frac{\lambda X_n}{\log n} = 1\right) = 1$

Contrast this conclusion with the claim that $EX_n = \frac{1}{\lambda}$:

as we just showed, X_n tends to congregate around $\frac{\log n}{\lambda}$!

Thus EX is generally not indicative of the typical behavior