ENEE620-23. Home assignment 1. Date due September 15, 11:59pm EDT.

Instructor: A. Barg

Please submit your work as a **single PDF file** to ELMS (under the "Assignments" tab)
• Papers submitted as multiple pictures of individual pages are difficult for grading and **will not be accepted**.
• Justification of solutions is required.
• Each problem is worth 10 points.

Problem 1. Let $(A_n)_n, (B_n)_n$ be sequences of subsets of Ω.

(a) Show that

$$\limsup_{n\to\infty}(A_n \cup B_n) = (\limsup_{n\to\infty} A_n) \cup (\limsup_{n\to\infty} B_n).$$

(b) If $A_1 \subset A_2 \subset \dots A_n \subset \dots$, what is $\limsup_{n\to\infty} A_n$? What is $\liminf_{n\to\infty} A_n$?

If $A_1 \supset A_2 \supset \dots A_n \supset \dots$, what is $\limsup_{n\to\infty} A_n$? What is $\liminf_{n\to\infty} A_n$?

Problem 2. Let $\Omega = \{1,2,3,4,5,6\}$ and $A = \{2,4\}, B = \{6\}$.

(a) What is the σ-algebra generated by the pair A, B on Ω?

(b) The σ-algebra generated by an RV X on the sample space Ω is defined as

$$\sigma(X) = \sigma(\{\omega : X(\omega) \le x\}, x \in \mathbb{R}),$$

i.e., the σ-algebra generated by all the subsets $\{\omega : X(\omega) \le x\}, x \in \mathbb{R}$.

Let $\Omega = \{1,2,3,4,5,6\}$. Consider a random variable on Ω given by $X(1) = 0.5, X(2) = 1, X(3) = 0.5, X(4) = X(5) = X(6) = 1.5$. Give an explicit description of $\sigma(X)$, i.e., list <u>all subsets</u> in $\sigma(X)$.

Problem 3. Consider the standard probability space on $\Omega = (0,1]$, i.e., the triple $(\Omega, \mathcal{B}(\Omega), \lambda)$, where $\mathcal{B}(\Omega)$ is the Borel σ-algebra and λ is the Lebesgue measure. Define the following random variables:

- $X_1(\omega) = 0$ for all $\omega \in \Omega$,
- $X_2(\omega) = \mathbb{1}_{\{0.2\}}(\omega) \quad (X_2(\omega) = 1$ if $\omega = 0.2$ and 0 otherwise),
- $X_3(\omega) = \mathbb{1}_{\mathbb{Q}}(\omega) \quad (X_3(\omega)$ =1 if ω is rational and 0 if it is not).

What is $\sigma(X_i)$ for $i = 1,2,3$, where the notation is explained in Problem 2b ?

Problem 4. (a) Show that if X, Y are RVs on some probability space, then $X + Y$ is also an RV on the same space.

(b) Let $X \sim \text{Unif}[0,2]$ (a uniform RV on $[0,2]$). Define $Y = \max\{1, X\}$ and $Z = \min\{X, X^2\}$. Show that Y and Z are RVs and find their distribution functions.

Problem 5. (How do exponential RVs behave in the long run?) Recall that the pdf of the exponential distribution $\text{Exp}(\lambda)$ is given by $f(x) = \lambda e^{-\lambda x}\mathbb{1}_{\{x\ge 0\}}$ for some $\lambda > 0$. Consider a sequence of independent exponential RVs $X_n, n \ge 1$, each of which has pdf $f(x)$.

(a) Given an $\epsilon > 0$ find

$$P\Big(\Big\{\omega : \frac{\lambda}{\log n} X_n(\omega) > 1 - \epsilon\Big\} \text{ i.o.}\Big).$$

(b) Given an $\epsilon > 0$ find

$$P\Big(\Big\{\omega : \frac{\lambda}{\log n} X_n(\omega) > 1 + \epsilon\Big\} \text{ i.o.}\Big).$$

(c) From the previous results, what is the probability of the event $\{\omega : \limsup_n \frac{\lambda X_n}{\log n} = 1\}$?