ENEE620. Final examination, 5/18/2021. Instructor: A. Barg

- Please submit your work to ELMS Assignments as a single PDF file by Wednesday, 5/19/21, 10:00am EDT.
- Each problem is 10 points. Max score=50 points
- Your answers should be justified. Giving just the answer may result in no credit for the problem.
- Please pay attention to the writing. You may lose points if your paper is difficult to read.

Problem 1.

Let $(N(t), t \geq 0)$ be a Poisson process with rate $\lambda = 2$.

(a) Let $T_k, k = 1, 2, \ldots$ be the kth arrival time. Find $P(T_1 + T_2 < T_3)$.

(b) Let $S \sim$ Unif$[0, 1]$ be independent of $N(t)$. Find $E[N^2(S)]$, where $N(S)$ is the number of arrivals at a random time S.

(c) Define $X_n = N(n^2), n = 0, 1, 2, \ldots$. Find the probability $P(X_{n+1} = j | X_n = i, X_{n-1} = i_{n-1}, \ldots, X_0 = i_0)$ and argue whether the sequence $(X_n, n \geq 0)$ forms a Markov chain.

Problem 2.

Let $(X_n)_n$ be a sequence of RVs, not assumed to be identically distributed or independent, and let $S_n = \sum_{i=1}^n X_n, n \geq 1$ be the partial sums.

(a) Show that $X_n \overset{\text{a.s.}}{\to} 0, n \to \infty$ implies that $\frac{S_n}{n} \overset{\text{a.s.}}{\to} 0$;
(b) Show that $X_n \overset{\text{a.s.}}{\to} 0, n \to \infty$ implies that $\frac{1}{\log n} \sum_{k=1}^n \frac{X_k}{k} \overset{\text{a.s.}}{\to} 0$;
(c) Show that $X_n \overset{\text{a.s.}}{\to} 0, n \to \infty$ implies that $\frac{1}{\log \log n} \sum_{k=1}^n \frac{X_k}{k \log k} \overset{\text{a.s.}}{\to} 0$.

(d) Show that $X_n \overset{\text{p}}{\to} 0, n \to \infty$ does not necessarily imply that $\frac{S_n}{n} \overset{\text{p}}{\to} 0$ as $n \to \infty$.

(e) Now assume that (X_n) are i.i.d. (do not assume that $X_n \to 0$). Show that as $n \to \infty$,

$$\frac{X_1 + X_2 + \cdots + X_n}{X_1^2 + X_2^2 + \cdots + X_n^2} \overset{\text{a.s.}}{\to} \frac{EX}{\text{Var}(X) + (EX)^2} = \frac{EX}{EX^2}.$$

Problem 3.

Let $(Y_n)_n$ be a sequence of i.i.d. RVs such that $P(Y = \frac{1}{2}) = P(Y = \frac{3}{2}) = \frac{1}{2}$, and set $X_n = \prod_{1 \leq i \leq n} Y_i, n \geq 1$.
(a) Show that $(X_n)_n$ forms a martingale; find EX.
(b) Show that the martingale converges and identify the limit. Does it converge in L^1? If yes, explain which sufficient conditions you used; if not, explain which necessary conditions fail to hold.

Problem 4.

A queueing system has 2 servers with exponential service rates μ_1 and μ_2. Customers enter the system at Poisson rate λ and join a single queue. If the arriving customer finds the system empty, he chooses one of the servers randomly with probability 1/2.
(a) Show that the system can be modeled by a Markov chain with states $\mathcal{S} = \{0, a, b, 2, 3, \ldots\}$, where a means that there is one customer at server 1 and b that there is one customer at server 2 (and no other customers in the system). Prove that this Markov chain is reversible, i.e., it satisfies the detailed balance condition $\pi_s Q_{st} = \pi_t Q_{ts}$ for all $t, s \in \mathcal{S}$.
(b) Find the limiting distribution π of the system.

Problem 5.

(a) Let $Y \sim$ Unif$(-1, 1)$ and consider a random process $X(t) = Y^3 t, t \geq 0$. Is the process $X(t)$ stationary?

(b) Now let $Y \sim$ Unif$(0, 1)$ and consider $X(t) = e^Y t, t \geq 0$. Are the increments of $X(t)$ independent? Are they stationary (meaning that for any $0 < t_1 < t_2$ the distribution of $X(t_2 + s) - X(t_1 + s)$ does not depend on s) ?