ENEE324, Home assignment 6. Date due April 12, 2026, 11:59pm EDT.

Instructor: Alexander Barg

Please upload your work as a **single PDF file** to ELMS (under the "Assignments" tab)

• Submissions on paper or by email will not be accepted.
• Please do not submit your solutions as multiple separate files (pictures of individual pages). Such submissions are difficult to grade and will not be accepted.

• Justification of solutions is required.
• Each problem is worth 10 points unless noted otherwise.

Problem 1 (Extending Prob.6 of HW5)**.** Let $X_1, \dots, X_n$ be independent random variables uniformly distributed on $(0, 1)$. Let

$$M = \max(X_1, \dots, X_n).$$

(1) Find the distribution function (cdf) and density (pdf) of M.
(2) Compute $\mathbb{E}[M]$.
(3) Let $G = 1 - M$ (the gap between 1 and the maximum). Compute $\mathbb{E}[G]$.
(4) In other words, $\mathbb{E}[G]$ is the expected distance from 1 to the closest point among n random points in $(0, 1)$. How does $E(G)$ behave as $n \to \infty$?

(1) $P(M \leq x) = P(\max(X_1, \dots, X_n) \leq x)$

$= (P(X_1 \leq x))^n \quad \because$ independence

$= x^n, \quad 0 \leq x \leq 1$

$$F_M(x) = \begin{cases} 0 & x \leq 0 \\ x^n & 0 \leq x \leq 1 \\ 1 & 1 \leq x \end{cases} ; \qquad f_M(x) = \begin{cases} 0 & x \leq 0 \\ n x^{n-1} & 0 \leq x \leq 1 \\ 0 & 1 \leq x \end{cases}$$

(2) $EM = \int_0^1 n x^n \, dx = \frac{n}{n+1}$

(3) $EG = E(1 - M) = 1 - \frac{n}{n+1} = \frac{1}{n+1}$

(4) $EG \to 0$ as $n \to \infty$, meaning that for many points, the expected gap of the rightmost point to 1 becomes smaller, approaching 0

Problem 2. Let $X \sim \mathcal{N}(0,1)$ be the standard normal random variable.

(1) Show that for any differentiable function g such that the expectations below exist,
$$\mathbb{E}[Xg(X)] = \mathbb{E}[g'(X)].$$
(*Hint:* Use integration by parts with the standard normal density.)

(2) Use part (1) with $g(x) = x$ to compute $\mathbb{E}[X^2]$.

(3) Use part (1) with $g(x) = x^2$ to compute $\mathbb{E}[X^3]$.

(4) Use part (1) with $g(x) = e^{tx}$ to compute $\mathbb{E}[e^{tX}]$ for $t \in \mathbb{R}$.

(1) $$\int_{-\infty}^{\infty} x\, g(x) \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}\, dx = -\underbrace{\frac{1}{\sqrt{2\pi}} g(x)}_{u} \underbrace{e^{-x^2/2}}_{\to 0} \Big|_{-\infty}^{\infty} + \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}} g'(x)\, dx$$

$$= E[g'(X)]$$

assuming $g(x)e^{-x^2/2} \to 0$ for $x \to \pm\infty$ as is the case in (2)-(4) below

(2) $$\int_{-\infty}^{\infty} x^2 \varphi(x)\, dx = \int_{-\infty}^{\infty} \varphi(x)\, dx = 1 \qquad (g(x) = x, \text{ so } g'(x) = 1)$$

$$\varphi(x) = e^{-x^2/2}/\sqrt{2\pi}$$

(3) $$\int_{-\infty}^{\infty} x^3 \varphi(x)\, dx = 2\int_{-\infty}^{\infty} x\, \varphi(x)\, dx = 2\, EX = 0 \qquad (g'(x) = 2x)$$

(4) Ee^{tX} can be computed in several ways. The problem suggests to use Pt. (1) with $g(x) = e^{tx}$, so $g'(x) = te^{tx}$

Let $M(t) = Ee^{tX}$

$$M'(t) = t\, M(t)$$

or $$\frac{M'(t)}{M(t)} = t \Rightarrow \ln M(t) = \frac{t^2}{2} + \text{const}.$$

$$M(0) = 1 \Rightarrow \text{const} = 0$$

$$\boxed{M(t) = e^{t^2/2}}$$

Here is another way which we will use later in class:

We have $-\frac{x^2}{2}+tx=-\frac{1}{2}\left(x^2-2tx+t^2-t^2\right)=-\frac{(x-t)^2}{2}+\frac{t^2}{2}$

$$M(t)=\frac{1}{\sqrt{2\pi}}\int e^{tx-\frac{x^2}{2}}\,dx = e^{\frac{t^2}{2}}\underbrace{\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{-\frac{(x-t)^2}{2}}\,dx}_{\text{pdf of } \mathcal{N}(t,1) \text{ integrates to } 1}=\boxed{e^{\frac{t^2}{2}}}$$

Problem 3. Let $X \sim \mathcal{N}(100,\, 10^2)$ represent the score of a student on a standardized test.

(1) What is the probability that a randomly chosen student scores above 120?
(2) Find the cutoff score c such that only the top 10% of students score above c.
(3) Suppose 5 students are selected independently. What is the probability that at least one of them scores above 120?
(4) Let ~~M~~ S be the ~~maximum~~ minimum score among the ~~5~~ 3 students. Compute ~~$\mathbb{P}(M \le 120)$~~. $P(S<80)$
(5) Approximate the expected number of students (out of 1000) who score above ~~130~~. 115.

(1) $P(X>120) = 1-\Phi\left(\frac{120-100}{10}\right) = 1-\Phi(2) \approx 0.0228$

(2) $1-\Phi\left(\frac{c-100}{10}\right) = 0.1 \implies c = 10\,\Phi^{-1}(0.9)+100 \approx 112.8$

(3) $P(\max(X_1,\dots,X_5)>120) = 1-(P(X_1\le 120))^5 = 1-(\Phi(2))^5 \approx 0.109$

(4) $P(\min(X_1,\dots,X_3)<80) = 1-(P(X_1\ge 80))^3 = 1-(1-\Phi(-2))^3 \approx 0.0667$

(5) Let $b = P(X>115) = 1-\Phi(1.5) \cong 0.0668$

Assuming that the scores are independent, the number of students in class with score >110 is $\mathrm{Binom}(1000, p)$

Its expected value is $1000p \approx 66.8$

Problem 4. A train departs from a station every 20 minutes starting at 6:00 a.m. A passenger arrives at the station at a random time uniformly distributed between 7:10 a.m. and 8:30 a.m.

(1) What is the probability that the passenger waits at most 5 minutes for the next train?
(2) What is the probability that the passenger waits at least 10 minutes?
(3) Let W denote the waiting time (in minutes). What is the distribution of the random variable W?
(4) Compute the expected waiting time $\mathbb{E}[W]$.

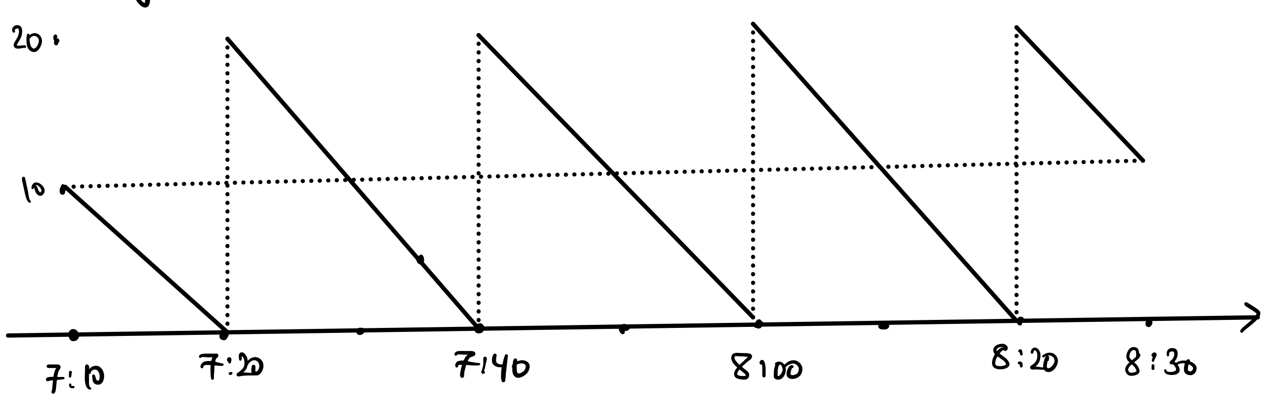
(1) The passenger's arrival interval is 80 min ; there are 4 possible trains for him to catch. There are four 5 min intervals for a ≤ 5 min wait. The probability of them is $\frac{20}{80} = \frac{1}{4}$

(2) Arguing similarly, we obtain

$$\frac{1}{80}\left(\{7{:}20 - 7{:}30\} \cup \{7{:}40 - 7{:}50\} \cup \{8{:}00 - 8{:}10\} \cup \{8{:}20 - 8{:}30\}\right) = \frac{1}{2}$$

Or: his possible arrival times are the 10-min intervals following any train departure, so they take ½ of the time line ; consult the plot in pt. (3)

(3) Range $(W) = [0,20]$



Let A_i, $i = 1, \ldots, 5$ be the 20-min arrival intervals (the 1st of them "wraps around")

$$P(W \leq x) \overset{\text{LOTP}}{=} \sum_{i=1}^{5} P(W \leq x \mid A_i) P(A_i) = \frac{1}{5}\left(5 \cdot P(W \leq x \mid A_1)\right) = \frac{x}{20}, \quad 0 \leq x \leq 20$$

$\therefore W \sim \text{Unif}(0,20)$

(4) $EW = 10$

Problem 5. Let $X \sim \text{Exp}(\lambda)$ (exponential RV with parameter λ) represent the waiting time in hours until a system experiences a failure.

Suppose that the system is observed only at integer times, and define

$$Y = \lfloor X \rfloor,$$

the integer part (floor) of X.

(1) Show that Y takes values in $\{0, 1, 2, \dots\}$ and compute

$$\mathbb{P}(Y = k), \quad k = 0, 1, 2, \dots.$$

(2) Show that Y has a geometric-type distribution and identify its parameter in terms of λ.
(3) Compute $\mathbb{E}[Y]$.
(4) Compute $\mathbb{P}(X - Y > t)$ for $0 \le t < 1$ and interpret this quantity.

(1) Y takes values $\{0,1,2,\dots\}$ by definition.

$$P(Y=k) = P(k \leq X < k+1) = F_X(k+1) - F_X(k) = e^{-\lambda k} - e^{-\lambda(k+1)}$$

$$= e^{-\lambda k}(1 - e^{-\lambda})$$

(2) Let $p = 1 - e^{-\lambda}$, then $P(Y=k) = p(1-p)^k$, $k = 0, 1, \dots$,

so $Y \sim \text{Geom}(p)$

(3) $EY = \frac{1-p}{p} = \frac{e^{-\lambda}}{1-e^{-\lambda}}$ as computed in class, Lecture 11

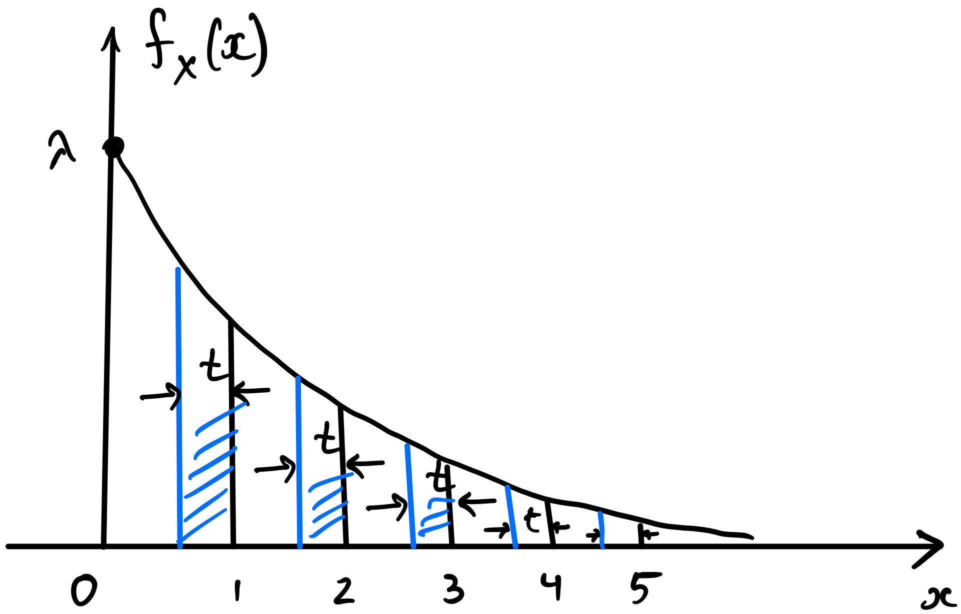
(4) By definition $0<t<1$.

$$P(X-Y>t) \overset{\text{LOTP}}{=} \sum_{k=0}^{\infty} P(X-k>t, Y=k) = \sum_{k=0}^{\infty} \underbrace{P(k+t<X<k+1)}_{\text{since } t<1}$$

$$= \sum_{k=0}^{\infty} \left[1 - e^{-\lambda(k+1)} - 1 + e^{-\lambda(k+t)}\right] = \sum_{k=0}^{\infty} (e^{-\lambda t} - e^{-\lambda})(e^{-\lambda})^k$$

$$= \frac{e^{-\lambda t} - e^{-\lambda}}{1 - e^{-\lambda}}$$

Observe that $e^{-\lambda t} - e^{-\lambda} = P(t < X < 1)$, which is the first of the infinite number of segments where $X - Y > t$. This segment has the highest probability, and the multiplier $\frac{1}{1-e^{-\lambda}} > 1$ is a correction factor that accounts for the other segments. This factor depends on λ, which controls the decay rate of the pdf $f_X(x) = \lambda e^{-\lambda x}$, $0 \leq x < \infty$



Problem 6. A system consists of n identical components operating independently. Each component fails after an exponential time with rate λ, i.e., its lifetime is $\mathrm{Exp}(\lambda)$. All components are installed at time 0.

Let T be the time at which the k-th failure occurs, where $1 \leq k \leq n$.

(1) For a fixed time $t > 0$, let $N(t)$ denote the number of components that have failed by time t. Show that $N(t) \sim \mathrm{Binomial}(n, 1 - e^{-\lambda t})$.
(2) Express the event $\{T \leq t\}$ in terms of $N(t)$.
(3) Use part (1) to compute $\mathbb{P}(T \leq t)$.
(4) Deduce the distribution function of T.
(5) Compute $\mathbb{E}[N(t)]$ and interpret it in terms of the expected number of failures by time t.

(1) $P(\text{a component fails by time } t) = P(\mathrm{Exp}(\lambda) < t) = F_{\mathrm{Exp}(\lambda)}(t) = 1 - e^{-\lambda t}$

Since they fail independently,

$$N(t) \sim \mathrm{Binom}(n, p), \quad \text{where } p = 1 - e^{-\lambda t}.$$

(2) If there are k failures by time t, then $N(t) \geq k$.

$$\{T \leq t\} = \{N(t) \geq k\}.$$

(3) Let $p = 1 - e^{-\lambda t}$. Then

$$P(T \leq t) = P(N(t) \geq k) = \sum_{j=k}^{n} \binom{n}{j} p^j (1-p)^{n-j}$$

(4) $F_T(t) = P(T \leq t)$

(5) $\mathbb{E}\, N(t) = np = n(1 - e^{-\lambda t})$

As we let t increase, $e^{-\lambda t} \to 0$, so $\mathbb{E} N(t) \to n$, meaning that after a long wait, on average all components fail