ENEE324, Home assignment 5. Date due April 5, 2026, 11:59pm EDT.

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Please upload your work as a **single PDF file** to ELMS (under the "Assignments" tab)

- Submissions on paper or by email will not be accepted.
- Please do not submit your solutions as multiple separate files (pictures of individual pages). Such submissions are difficult to grade and will not be accepted.

- Justification of solutions is required.
- Each problem is worth 10 points unless noted otherwise.

Problem 1. There is a group of n people.

(a) Fix a number k, $1 \le k \le n$. Find the expected number of days in a 365-day calendar year such that on each of these days exactly k people out of the group have their birthday.

(b) Find the expected number of days in a 365-day calendar year such that on each of them, at least 2 people out of this group have their birthday.

(a) Probability that k people have their birthday on the same day $= \left(\frac{1}{365}\right)^k \left(\frac{364}{365}\right)^{n-k}$

There are $\binom{n}{k}$ ways to select such group of people.

Ans: $n\binom{n}{k}\left(\frac{1}{365}\right)^k\left(\frac{364}{365}\right)^{n-k}$

(b) Let $p = \text{Prob}(\geq 2 \text{ people have their birthday on the same day})$

$$p = 1-\left(\frac{364}{365}\right)^n - \frac{n}{365}\left(\frac{364}{365}\right)^{n-1} = 1-\left(\frac{364}{365}\right)^{n-1}\left(\frac{364-n}{365}\right)$$

$= \mathbb{E}(\text{indicator } I(\leq 1 \text{ birthdays on a given day}))$

Use the fundamental bridge to conclude that

$$\mathbb{E}(\#\text{ of days}) = 365p.$$

Problem 2. Let X be a discrete random variable with values in $\{0, 1, 2, \dots\}$ and finite variance. Define

$$g(X) = (-1)^X.$$

(1) Show that
$$\mathrm{Var}(g(X)) = 1 - \left(\mathbb{E}[(-1)^X]\right)^2.$$

(2) Suppose $X \sim \mathrm{FS}(p)$ (First Success) with
$$P(X = k) = p(1-p)^{k-1}, \qquad k = 1, 2, \dots.$$
Compute $\mathbb{E}[(-1)^X]$.

(3) Deduce a closed-form expression for $\mathrm{Var}((-1)^X)$.

(4) Evaluate $\mathrm{Var}((-1)^X)$ in the cases $p \to 0$ and $p \to 1$, and briefly interpret the result.

(1) $g(X)$ takes values ± 1, so $EX = \sum_{k=0}^{\infty} (-1)^k p_X(k)$

$E(g(X)^2) = \sum (g(X))^2 p_X(k) = 1$

$\mathrm{Var}(g(X)) = E(g(x))^2 - (EX)^2 = 1 - (EX)^2$

(2) For $X \sim FS(p)$, let $Z = g(X)$.

$E(Z) = \sum_{k=1}^{\infty} (-1)^k p(1-p)^{k-1} = (-p) \sum_{k=1}^{\infty} (-(1-p))^{k-1} = \frac{-p}{2-p}$

(3) $\mathrm{Var}(Z) = 1 - \left(\frac{p}{2-p}\right)^2$

(4) If $p \to 0$, then $EZ \nearrow 0$, $\mathrm{Var}(Z) \to 1$. The partial sums

$\sum_{k=1}^{t} (-1)^k p(1-p)^{k-1}$, $t = 1, 2, 3, \dots$

oscillate between near $(-p)$ and near 0, approaching $-\frac{p}{2-p}$.

(5) If $p \to 1$, then $EZ \searrow -1$, $\mathrm{Var}(Z) \to 0$. With high probability $X = 1$, i.e., $Z = -1$, approaching a constant.

Problem 3. Let $X \sim \text{Poisson}(\lambda)$.

(1) Show that
$$\mathbb{E}[X(X-1)] = \lambda^2.$$
(2) Use this to compute $\text{Var}(X)$.
(3) Let $Y = \mathbf{1}_{\{X \text{ is even}\}}$ be the indicator RV of the event $\{X \text{ is even}\}$. Compute $\mathbb{E}[Y]$.
(4) (Thinning) Suppose that each event counted by X is independently kept with probability p, and discarded otherwise. Let Z be the number of kept events. Show that $Z \sim \text{Poisson}(p\lambda)$.

(1) $P(X=k) = \frac{\lambda^k}{k!} e^{-\lambda}, \quad k = 0, 1, \ldots$

$$\mathbb{E}(X(X-1)) = EX^2 - EX = \lambda(1+\lambda) - \lambda = \lambda^2$$
[as computed in Lecture 14]

(2) $$\text{Var}(X) = E(X^2) - (EX)^2 = E(X(X-1)) + EX - (EX)^2$$
$$= \lambda^2 + \lambda - \lambda^2 = \lambda$$

(3) $$EY = \sum_{k=0}^{\infty} 1 \cdot \frac{\lambda^{2k}}{(2k)!} e^{-\lambda}$$ (summing over all even integers $k = 0, 2, 4, \ldots$)

$$= e^{-\lambda} \sum_{k=0}^{\infty} \frac{\lambda^{2k}}{(2k)!}$$

Recall $e^{\lambda} = 1 + \lambda + \frac{\lambda^2}{2!} + \frac{\lambda^3}{3!} + \ldots$; $e^{-\lambda} = 1 - \lambda + \frac{\lambda^2}{2!} - \frac{\lambda^3}{3!} + \ldots$, so

$$e^{\lambda} + e^{-\lambda} = 2\left(1 + \frac{\lambda^2}{2!} + \frac{\lambda^4}{4!} + \ldots\right) = 2 \sum_{k=0}^{\infty} \frac{\lambda^{2k}}{(2k)!}$$

$$\therefore EY = e^{-\lambda} \frac{e^{\lambda} + e^{-\lambda}}{2} = \frac{1 + e^{-2\lambda}}{2}$$

(4) Let Z be the # of events retained after the thinning.

$$P(Z=k) = P(Z=k \mid X \geq k) = \sum_{m=k}^{\infty} P(Z=k \mid X=m)$$

$$= \sum_{m=k}^{\infty} \binom{m}{k} p^k (1-p)^{m-k} \cdot P(X=m) = \frac{p^k}{(1-p)^k} \sum_{m=k} \binom{m}{k} (1-p)^m \frac{\lambda^m}{m!} e^{-\lambda}$$

$$= \frac{p^k}{(1-p)^k} e^{-\lambda} \sum_{m \geqslant k} \frac{((1-p)\lambda)^m}{k!(m-k)!} = \frac{p^k}{(1-p)^k} \frac{e^{-\lambda}}{k!} \sum_{m \geqslant k} \frac{((1-p)\lambda)^m}{(m-k)!}$$

$$= \frac{((1-p)\lambda)^k p^k}{(1-p)^k} \frac{e^{-\lambda}}{k!} \sum_{m=0}^{\infty} \frac{((1-p)\lambda)^m}{m!} = (\lambda p)^k \frac{e^{-\lambda}}{k!} \cdot e^{(1-p)\lambda} = \frac{(\lambda p)^k}{k!} e^{-\lambda p}$$

$\sim Poi(\lambda p)$

Problem 4. A sequence of independent Bernoulli random variables $X_1, X_2, \ldots, X_n$ is given, where

$$\mathbb{P}(X_i = 1) = \frac{1}{n}, \qquad \mathbb{P}(X_i = 0) = 1 - \frac{1}{n}.$$

Let

$$S_n = X_1 + X_2 + \cdots + X_n.$$

(1) Compute $\mathbb{E}[S_n]$.
(2) Compute $\mathrm{Var}(S_n)$.
(3) Show that for each fixed k,

$$\mathbb{P}(S_n = k) = \binom{n}{k}\left(\frac{1}{n}\right)^k\left(1 - \frac{1}{n}\right)^{n-k}.$$

(4) Show that for each fixed k,

$$\mathbb{P}(S_n = k) \longrightarrow \frac{e^{-1}}{k!} \quad \text{as } n \to \infty.$$

(5) Conclude that for large n, S_n is approximately $\mathrm{Poisson}(1)$, and estimate $\mathbb{P}(S_n = 0)$.

(1) $E S_n = n\, E X_1 = n \cdot \frac{1}{n} = 1$

(2) Start with $E X_1^2 = 1^2 \cdot \frac{1}{n} + 0^2\left(1 - \frac{1}{n}\right) = \frac{1}{n}$

$$E S_n^2 = E\left(\sum_{i=1}^{n} X_i\right)^2 = \sum_{i=1}^{n} E X_i^2 + 2 \sum_{i<j} E(X_i X_j)$$

We have $P(X_i X_j = 1) = \frac{1}{n^2}$ and $P(X_i X_j = 0) = 1 - \frac{1}{n^2}$, so $E(X_i X_j) = \frac{1}{n^2}$

$$E S_n^2 = 1 + 2\binom{n}{2} \cdot \frac{1}{n^2} = 1 + \frac{n(n-1)}{n^2} = 1 + \frac{n-1}{n} = \frac{2n-1}{n}$$

$$\mathrm{Var}(S_n) = \frac{2n-1}{n} - 1 = \frac{n-1}{n}$$

(3) Since $S_n \sim \mathrm{Binom}(n, 1/n)$ (the number of 1's in a sequence of independent Bernoulli RVs), we have $P(S_n = k) = \binom{n}{k}\left(\frac{1}{n}\right)^k\left(1 - \frac{1}{n}\right)^{n-k}$

(4) We know from L. 14 that if $\lambda = \frac{p}{n}$ is constant as $n \to \infty$, then the Binomial distribution $\to$ Poisson (λ). In our case $\lambda = 1$, which implies the claim

(5) $\therefore\ P(S_n = 0) \simeq \frac{1}{e} \approx 0.368$

Problem 5. Let $X \sim \text{Uniform}(0, 1)$.

(1) Compute $\mathbb{E}[X]$ and $\text{Var}(X)$.
(2) Let $Y = -\ln X$. Find the probability density function of Y .
(3) Compute $\mathbb{E}[Y]$.

(1) $EX = \int_0^1 x\,dx = \frac{1}{2}$; $EX^2 = \int_0^1 x^2\,dx = \frac{1}{3}$; $Var(X) = \frac{1}{3} - \frac{1}{4} = \frac{1}{12}$

(2) $F_Y(t) = P(-\ln X \le t) = P(X \ge e^{-t}) = 1 - e^{-t}, \quad 0 \le t < \infty$

$f_Y(t) = (F_Y(t))'_t = e^{-t}, \quad 0 \le t < \infty$

(3) $EY = \int_0^\infty t e^{-t}\,dt = -te^{-t}\Big|_0^\infty + \int_0^\infty e^{-t}\,dt = 1$

Problem 6. Let X_1, X_2 be independent random variables, each uniformly distributed on $(0,1)$. Let
$$M = \max(X_1, X_2), \qquad m = \min(X_1, X_2).$$

(1) Find the probability density functions of M and m.
(2) Compute $\mathbb{E}[M]$.
(3) Compute $\mathbb{E}[m]$.
(4) Compute $\mathbb{P}(X_1 + X_2 \le 1)$.

(1) $F_M(x) = P(X_1 \le x, X_2 \le x) = x^2, \quad 0 \le x \le 1$

$F_m(x) = P(\min(X_1, X_2) \le x) = 1 - P(X_1 > x, X_2 > x) = 1 - (1-x)^2 = 2x - x^2$
$0 \le x \le 1$

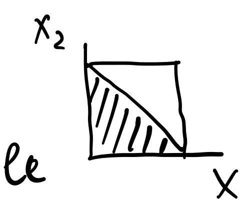
$f_M(x) = 2x$ if $0 \le x \le 1$ and 0 o/w

$f_m(x) = 2(1-x)$ if $0 \le x \le 1$ and 0 o/w

(2) $EM = \int_0^1 2x^2\,dx = \frac{2}{3}$

(3) $Em = \int_0^1 2x(1-x)\,dx = 1 - \frac{2}{3} = \frac{1}{3}$

(4) $P(X_1 + X_2 \le 1) = \frac{1}{2}$ b/c. it is the area of the hashed triangle



or:
$P(X_1 + X_2 \le 1) = \int_0^1 P(X_1 \le 1-x) f_{X_2}(x)\,dx = \int_0^1 x\,dx = \frac{1}{2}$