Derivation of the drift-kinetic equation

Introduction

The “drift-kinetic equation” is the basis for all calculations of neoclassical transport and flows, as well as the bootstrap current. There are several variants of the equation; one standard form is

\[ f_1 \cdot \nabla \tilde{f}_1 + v_d \cdot \nabla f_0 - \frac{Ze}{T} E_{||} v_|| f_0 = C \left\{ f_1 \right\} \] (1)

where \( f_0 \) is the leading-order Maxwellian distribution function, \( \tilde{f}_1 \) is the gyroaveraged perturbed distribution function, \( b = B / B, \) \( B = |B|, \) \( \nabla = \nabla_{gg} \) is the sum of magnetic, \( E \times B, \) and parallel drifts, \( \Omega = ZeB / (mc) \) is the gyrofrequency, and \( \kappa = b \cdot \nabla b \) is the field curvature. The independent variables (which are held fixed in the gradients in (1)) are the magnetic moment \( \mu = v_1^2 / (2B) \) and leading-order total energy \( W = v^2 / 2 + Ze \Phi_0, \) where \( \Phi_0 \) is the leading-order electrostatic potential.

Several variations of the equation are possible. Often the parallel drift in (2) is dropped. Sometimes the \( E_{||} \) term in (1) is written \( \left\{ f_{||} \right\} = \frac{Ze}{T} E_{||} v_{||} \frac{\partial f_0}{\partial W}. \)

Orderings:

The drift-kinetic equation is derived from the Fokker-Planck equation by expanding in the small parameter \( \rho_\ast = \rho / L = v_{bh} / (\Omega L) \) where \( \rho \) is the thermal gyroradius, and \( L \) is the scale length for variation in all quantities: \( B, f_0, f_1, \) and \( \Phi. \) This is in contrast to gyrokinetics, in which \( f_1 \) and \( \Phi_1 \) are permitted to vary on a scale length comparable to \( \rho. \) The collision frequency \( \nu \) is ordered as \( \nu \sim \rho_\ast \Omega. \) The electric field is taken to be electrostatic to leading order: \( E = -\nabla \Phi_0 + E_\ast \) where \( E^* \sim \rho_\ast E, \) and the leading-order electric field \( -\nabla \Phi_0 \) is ordered using \( v_{E \times B} \sim \rho \nu v_{bh}. \) Time derivatives are taken to be small: \( \partial / \partial t \sim \rho_\ast^2 \Omega. \)

Derivation

Begin with the Fokker-Planck equation \( Df = C \left\{ f \right\} \) where

\[ D = \left[ \frac{\partial f}{\partial t} \right]_v + V \cdot \nabla f + \frac{Ze}{m} \left( E + \frac{1}{c} v \times B \right) \cdot \nabla v f. \] (3)

Subscripts on partial derivatives indicate quantities that are held fixed in differentiation.

We introduce cylindrical velocity-space coordinates \( \left( v_{\perp}, \varphi, v_{||} \right) \) so that \( v = v_{||} b + v_{\perp} \) where

\[ v_{\perp} = v_{\perp} (e_0 \cos \varphi + e_1 \sin \varphi), \] (4)

\( e_1 \) and \( e_2 \) are position-dependent unit vectors orthogonal to \( B, \) and \( \varphi \) is the gyrophase. The system \( (e_1, e_2, b) \) is right handed. A brief calculation gives
\[
V_{\nu}Q = b \left( \frac{\partial Q}{\partial v_{\parallel}} \right)_{v_{\perp}, \varphi} + \nu_{\perp} \left( \frac{\partial Q}{\partial v_{\parallel}} \right)_{v_{\perp}, \varphi} + \frac{1}{\nu_{\perp}^2} b \times \nu_{\perp} \left( \frac{\partial Q}{\partial \varphi} \right)_{v_{\perp}, \varphi}
\]

(5)

for any quantity \( Q \). We next introduce

\[
\mu = \nu_{\perp}^2 / (2B) \quad \text{and} \quad W = \nu_{\parallel}^2 / 2 + Ze\Phi_0 / m
\]

(6)

where \( \nu_{\parallel}^2 = \nu_{\perp}^2 + \nu_{\parallel}^2 \). A bit of algebra gives

\[
DW = (Ze / m) E^* \cdot v
\]

(7)

where \( E^* = E + \nabla \Phi_0 \),

\[
D\mu = -\frac{\mu}{B} v \cdot \nabla B - \frac{\nu_{\parallel} \nu_{\perp} (\nabla b) \cdot (\nabla b)}{B} + \frac{Ze}{mB} E \cdot v_{\perp},
\]

(8)

and

\[
D\varphi = -\Omega + G
\]

(9)

where \( G \) is an ugly bunch of terms of order \( \rho \Omega \) (arising from \( (\nabla \varphi)_{\parallel} \)). Thus, the Fokker-Planck equation can be written

\[
\frac{\partial f}{\partial t} + v \cdot \nabla f + (DW) \frac{\partial f}{\partial W} + (D\mu) \frac{\partial f}{\partial \mu} + (D\varphi) \frac{\partial f}{\partial \varphi} = C \{ f \}.
\]

(10)

Here, and for the rest of the calculation, partial derivatives hold \( \mu, W, \) and \( \varphi \) fixed.

We now introduce the gyroaveraging operation \( \bar{Q} = (2\pi)^{-1} \int_0^{2\pi} Q \, d\varphi \) where position, \( W, \) and \( \mu \) are held fixed in the integration. Notice

\[
\bar{DW} = (Ze / m) E_{\parallel}^* / v_{\parallel}.
\]

(11)

To compute \( \bar{D\mu} \), we use

\[
\nu_{\parallel} \nu_{\perp} = \nu_{\parallel}^2 b \cdot b^2 / 2 (I - bb)
\]

(12)

and \( bb \cdot \nabla b = 0 \) to obtain \( \bar{D\mu} = 0 \). Introducing \( \tilde{f} = f - \bar{f} \), it will turn out to be convenient to write the Fokker-Planck equation as

\[
\frac{\partial \tilde{f}}{\partial t} + v \cdot \nabla \tilde{f} + (DW) \frac{\partial \tilde{f}}{\partial W} + (D\mu) \frac{\partial \tilde{f}}{\partial \mu} + D\tilde{f} = C \{ \tilde{f} + \bar{f} \}.
\]

(13)

Applying a gyroaverage,

\[
\frac{\partial \tilde{f}}{\partial t} + v_{\parallel} \bar{b} \cdot \nabla \tilde{f} + \frac{Ze}{m} E_{\parallel}^* \nu_{\parallel} \frac{\partial \tilde{f}}{\partial W} + D \tilde{f} = C \{ \tilde{f} + \bar{f} \}.
\]

(14)

Subtracting this result from (13) gives

\[
\nu_{\perp} \cdot \nabla \tilde{f} + \frac{Ze}{m} E_{\perp}^* \cdot \nu_{\perp} \frac{\partial \tilde{f}}{\partial W} + (D\mu) \frac{\partial \tilde{f}}{\partial \mu} + D\tilde{f} = C \{ \tilde{f} + \bar{f} \} - C \{ \tilde{f} + \bar{f} \}.
\]

(15)

Let us now begin to apply the ordering assumptions given above. The leading term in (10) is

\( -\Omega \frac{\partial \tilde{f}_0}{\partial \varphi} = 0 \) from the \( D\varphi \) term, so \( \tilde{f}_0 = 0 \), and \( \bar{f} \sim \rho \Omega \bar{f} \). We henceforth drop the overbar on \( f_0 \).

Next, the leading terms in (14) are the \( O(\rho \Omega \bar{f}) \) terms

\[
v_{\parallel} \bar{b} \cdot \nabla f_0 = C \{ f_0 \}.
\]

(16)
At this point, a rigorous derivation can be given to show $f_0$ must be a Maxwellian. For simplicity we will not give this derivation here. If $f_0$ is Maxwellian, then $C\{f_0\} = 0$, so (16) becomes $\nu b \cdot \nabla f_0 = 0$. Also, we may linearize the collision operator and use $C_0\{g\} = C_0\{g\}$ to simplify the right-hand side of (14) to $C\{f\}$.

Now consider the $O(\rho_0 f_0^2)$ terms in (15):

$$v_\perp \cdot \nabla f_0 - \Omega \frac{\partial f_0}{\partial \varphi} = 0.$$  \hspace{1cm} (17)

Using

$$v_\perp = \frac{\partial}{\partial \varphi} (v \times b)$$ \hspace{1cm} (18)

then (17) may be integrated to obtain

$$f_1 = -\rho \cdot \nabla f_0$$ \hspace{1cm} (19)

where

$$\rho = \Omega^{-1}b \times v.$$ \hspace{1cm} (20)

We now form the drift-kinetic equation from the $O(\rho_0 f_0^2)$ terms in (14):

$$\nu b \cdot \nabla f_1 - \frac{Ze}{T} E^* \nu_0 f_0 + Df_1 = C\{f_1\}. \hspace{1cm} (21)$$

We must evaluate

$$Df_1 = -D[\rho \cdot \nabla f_0] = -(\frac{D\rho}{\chi}) \cdot \nabla f_0 - \frac{D(Df_0)}{\gamma} \cdot \nabla f_0.$$ \hspace{1cm} (22)

We can drop the time derivative in $D$ since it is high order. First consider the term $Y$, writing

$$D(\nabla f_0) = \left[ v \cdot \nabla + (DW) \frac{\partial}{\partial W} \right] \nabla f_0 = v \cdot \nabla \nabla f_0 + \frac{Ze}{m} E^* \cdot v \cdot \frac{\partial}{\partial W} \nabla f_0.$$ \hspace{1cm} (23)

The $E^*$ term is higher order than the others in (22), so it can be neglected. Then $Y = \rho v \cdot \nabla \nabla f_0$. We find

$$\overline{\rho v} = \frac{v_0^2}{2\Omega} (e_2 e_1 - e_1 e_2)$$ \hspace{1cm} (24)

to be antisymmetric, so since $\nabla \nabla f_0$ is symmetric, $Y = 0$. We can evaluate $X$ using (3), finding

$$Dp = v \cdot \nabla \left( \frac{1}{\Omega} b \right) \times v - \frac{c}{B} E \times b.$$ \hspace{1cm} (25)

Gyroaveraging,

$$\overline{Dp} = \left( \frac{v_0^2}{2} - \frac{v_0^2}{2} \right) \cdot b \cdot \nabla \left( \frac{1}{\Omega} b \right) \times b + \frac{v_0^2}{2} \sum_{i=1}^3 e_i \cdot \nabla \left( \frac{1}{\Omega} b \right) \times e_i - \frac{c}{B} E \times b$$

$$= \left( \frac{v_0^2}{2} - \frac{v_0^2}{2} \right) b \times \nabla \left( \frac{1}{\Omega} b \right) - \frac{v_0^2}{2\Omega} b \cdot \nabla B - \frac{c}{B} E \times b$$ \hspace{1cm} (26)

where $k = b \cdot \nabla b$. Then applying

$$\nabla \times b = bb \cdot \nabla \times b - k \times b,$$ \hspace{1cm} (27)

we obtain $\overline{Dp} = -v_d$ where $v_d$ is given in (2). Thus, (21) becomes
\[ u_\parallel \mathbf{b} \cdot \nabla f_1 = \frac{Z_e}{T} E_\parallel^* f_0 + v_d \cdot \nabla f_0 = C \left\{ f_1 \right\}. \] (28)

Taking \( \mathbf{b} \cdot \nabla \Phi_0 = 0 \) so \( E_\parallel^* = E_\parallel \), we obtain the desired result (1), concluding the proof.