Another Boozer-coordinate-free motivation for quasisymmetry

We would like any confinement device to have the following omnigenity property: the $\psi$ coordinate of each trapped particle has no net change per bounce. Equivalently, $\Delta \psi = 0$ where

$$\Delta \psi = \psi_{\text{final}} - \psi_{\text{initial}} = \oint \mathbf{v} \cdot \nabla \psi \, dt$$

(1)

Here, $\oint$ indicates an integral over a bounce, holding the magnetic moment $\mu = m v^2 / (2 B)$ and energy $m v^2 / 2$ fixed. Rather than integrate over the actual orbit (which has a nonzero width), we take (1) to be an integral along a field line, which is the same aside from a correction of order $\rho$, and which is much easier to work with mathematically.

Before considering stellarators, let us examine in detail why (1) is satisfied in a tokamak. First, we notice

$$\Delta \psi = \oint \mathbf{v} \cdot \nabla \psi \, dt = 2 \int_{\ell_{-}}^{\ell_{+}} (\mathbf{v} \cdot \nabla \psi) \frac{d\ell}{v_i}$$

(2)

where $\ell_{-}$ and $\ell_{+}$ are the two bounce points. Changing the variable of integration to $B = |B|$, we have

$$\Delta \psi = 2 \sum_{\gamma} \int_{B_{\text{min}}}^{B_{\text{tr}}^\gamma} (\mathbf{v} \cdot \nabla \psi) \frac{dB}{v_i |\mathbf{b} \cdot \nabla B|} = 2 \sum_{\gamma} \int_{B_{\text{min}}}^{B_{\text{tr}}^\gamma} (\mathbf{v} \cdot \nabla \psi) \frac{dB}{v_i |\mathbf{b} \cdot \nabla B|}$$

(3)

where $B_{\text{tr}} = m v^2 / (2 \mu)$ is the value of $B$ at which the particle is trapped, and where $\gamma = \text{sign} (\mathbf{b} \cdot \nabla B)$ is the "branch": $\gamma = +1$ if our location relative to $B_{\text{min}}$ is parallel to $\mathbf{B}$, and $\gamma = -1$ if our location relative to $B_{\text{min}}$ is antiparallel to $\mathbf{B}$. The sum over $\gamma$ is needed because each value of $B$ could correspond to one of two different $\ell$, as shown by the following figure. As with any change of variables, the absolute value of the Jacobian arises, and in the case of (3) this is $1 / |\mathbf{b} \cdot \nabla B|$.

Next, using $\mathbf{B} \times \nabla \psi = B^{-1} \mathbf{B} \times \nabla B \cdot \nabla \psi$ (which is true for any MHD equilibrium), we obtain
\[ \Delta \psi = \frac{2mc}{Ze} \sum_\gamma \int_{r_{s_{\min}}}^{r_{s_{\max}}} \left( \frac{v_\parallel^2 + v_\perp^2}{2} \right) \frac{1}{|v_\parallel| B^2} \frac{B \times \nabla B \cdot \nabla \psi}{B \cdot \nabla B} dB. \]  

(4)

In an axisymmetric MHD equilibrium, \( \mathbf{B} = \nabla \phi \times \nabla \psi + F(\psi) \nabla \phi \) where \( \phi \) is the toroidal angle and \( F(\psi) = RB_{\text{tor}} \) is a flux function. Therefore \( \frac{B \times \nabla \psi}{B \cdot \nabla B} = \frac{F(\psi) - R^2 B^2 \nabla^2 \phi}{B \cdot \nabla B} \), so

\[ \frac{B \times \nabla B \cdot \nabla \psi}{B \cdot \nabla B} = -F(\psi). \]  

(5)

Then we can pull this factor outside the integral in (4):

\[ \Delta \psi = -\frac{2mcF}{Ze} \sum_\gamma \int_{r_{s_{\min}}}^{r_{s_{\max}}} \left( \frac{v_\parallel^2 + v_\perp^2}{2} \right) \frac{1}{|v_\parallel| B^2} dB. \]  

(6)

It turns out that the remaining integral can actually be done, using the relation \( v^2 = v_\parallel^2 + (2\mu B / m) \) and recalling that \( \mu \) and \( \nu \) are fixed. However, we do not even need to do the integral, because the result evidently cannot depend on the branch (only on \( B_{\min}, B_{\max}, \nu, \) and \( \mu \)), and so the result will vanish in the \( \gamma \) sum. Physically, the radial drift in the \( \gamma > 0 \) branch is equal and opposite to the radial drift in the \( \gamma < 0 \) branch.

In the above argument, we only used axisymmetry in one small way: writing the left-hand side of (5) as a flux function, so that function could be pulled outside the integral. It did not even matter that the ratio equaled \( RB_{\text{tor}} \). In fact, it did not even matter that the ratio was a flux function – only that it was branch-independent! Thus, if we can find any nonaxisymmetric field in which

\[ \frac{B \times \nabla B \cdot \nabla \psi}{B \cdot \nabla B} = Y(\psi, B) \]  

(7)

for some function \( Y \), then (4) would become

\[ \Delta \psi = \frac{2mc}{Ze} \sum_\gamma L(B_{\min}, \nu, \mu) \]  

(8)

where

\[ L(B_{\min}, \nu, \mu) = \int_{r_{s_{\min}}}^{r_{s_{\max}}} \left( \frac{v_\parallel^2 + v_\perp^2}{2} \right) \frac{1}{|v_\parallel| B^2} Y(\psi, B) dB. \]  

(9)

Then the sum over \( \gamma \) in (8) immediately vanishes because \( L \) is independent of \( \gamma \). Thus, any magnetic field with the property (7) is omnigenous: \( \Delta \psi = 0 \).

As shown in Ref. [1], the quasisymmetry condition \( B = B(\psi, M \theta - N \zeta) \) for any integers \( M \) and \( N \) is satisfied if and only if

\[ \frac{B \times \nabla B \cdot \nabla \psi}{B \cdot \nabla B} = Y(\psi) \]  

(10)

This condition, that the ratio is a flux function, is a stricter condition than (7). Therefore quasisymmetry implies omnigenity.

I'm not sure if (7) is as general as omnigenity: here we have only proved that (7) is sufficient for omnigenity, not that (7) is necessary.