Problem  
Find \( y(x) \) if
\[
\frac{d}{dx} \left( p \frac{dy}{dx} \right) + q y = S
\]
on \( x = [a, b] \), \( y(a), y'(a) \) given
and \( y(b), y'(b) \) given

Where \( p = p(x) \), \( q = q(x) \), \( S = S(x) \), given

Set up of Green function \( G(x', x) \)

let \( y(x) = \int_a^b \delta(x'-x) y(x') \) 

let \( L G = \delta(x'-x) \)

where \( \delta' = \frac{d}{dx} \left( p \frac{d}{dx} \right) + q(x) \)
where, for the moment, b.c.'s on 6 at a or b are not specified.

Insert \( r_2 \rightarrow r_1 \) and integrate by parts twice =)

\[
y(x) = \int_a^b dx' \left[ \frac{d}{dx'} \left( \frac{p}{dx} \frac{dL}{dx} \right) + qL \right] y(x')
\]

by parts

\[
y(x) = \int_a^b dx' G(x',x) S(x')
\]

\[
+ \left[ p \left( \frac{dL}{dx} y - L \frac{dy}{dx} \right) \right]_a^b \text{ using integration by parts and } \tag{4}
\]

We now use the "freedom" in the b.c.'s for 6 to cast 4 into a form that will yield \( y(x) \). Suppose we are given Dirichlet boundary conditions, i.e., \( y(a) = y_a, \ y(b) = y_b \). Then, in the boundary \( \int_a^b \) term, we know what to use for \( y(a) \) and \( y(b) \), but we don't know \( \frac{dy}{dx}(a) \) or \( \frac{dy}{dx}(b) \), a priori.
In that case, if we use the boundary conditions $G(a) = 0$ and $G(b) = 0$ for $G$, the $\int_a^b$ term becomes $\left[ p \left( \frac{dG}{dx} \right) \right]_a^b$

$- \left[ p \left( \frac{dG}{dx} \right) \right]_a^b \frac{\partial}{\partial x_i} y_a$. Consequently, for Dirichlet boundary conditions

$y(a) = y_a, \quad y(b) = y_b$,

the solution for $g(x)$ is

$$y(x) = \int_a^b \ dx' \ 6(x',x) \ S(x')$$

$$+ p(b) G(b) y_b - p(a) \left( \frac{dG}{dx} \right) (a) y_a$$

where

$$\frac{d}{dx'} \left( p \frac{dG}{dx'} \right) + 2G = S(x' - x)$$

$$G(a) = 0, \quad G(b) = 0$$

Solution for $g(x)$ it can build $G$.
Alternatively, if we were given Neumann b.c.'s, \( y'(a) = y'_a, \ y'(b) = y'_b \), then, in the \( \int \lambda^b_a \) expression, we know the \( (dy/dx') \) terms but not the \( y \) terms. In that case, use the freedom in \( 6 \) b.c.'s to set \( \frac{d\lambda}{dx'}(b) = 0, \ \frac{d\lambda}{dx'}(a) = 0 \).

Thus, for Neumann b.c.'s

\[
\begin{align*}
\frac{d\lambda}{dx'}(b) &= 0, \\
\frac{d\lambda}{dx'}(a) &= 0.
\end{align*}
\]

The solution is

\[
y(x) = \int_a^b dx' \, 6(x',x) \, \lambda(x') = \int_a^b dx' \, 6(x',x) \, \lambda(x') - p(b) \, 6(b) \, y'_b + p(a) \, 6(a) \, y'_a
\]

\[\text{where} \]

\[
\begin{align*}
\frac{d\lambda}{dx'}(p \, \frac{d\lambda}{dx'}) + \lambda \, 6 &= \delta(x' - x) \\
\frac{d\lambda}{dx'}(a) &= 0, \ \frac{d\lambda}{dx'}(b) &= 0.
\end{align*}
\]

Solution for Neumann problem, it can find
Clearly, can generalize to "mixed" Dirichlet + Neumann b.c.'s. Later on, we will show other b.c.'s, such as "outgoing waves."

How do we find \( \alpha \)?

We try a "mixed" b.c. problem for a specific operator, as an example.

**Problem**: Suppose \( \frac{d^2y}{dx^2} + k^2 y = S(x) \), \( k > 0 \) given, \( S(x) \) given. Suppose \( y'(0) = 0 \), \( y(b) = 1 \). Find \( y(x) \), \( x = [0, b] \).

The Green function that goes with this \( \Delta \) and the specific mixed b.c.'s is
\[ \frac{d^2 G}{dx^2} - k^2 G = \delta(x'-x) \quad (7) \]

\[ \frac{dG}{dx}(0) = 0, \quad G(b) = 0. \]

\[ \text{Schematic:} \]

\[ \text{Green} \]

From (7), can show by \[ \int_0^x \frac{dx'}{x' - \epsilon} \quad \epsilon \to 0, \quad x - \epsilon \]

that \[ \left[ \frac{dG}{dx} \right]_{x' = x} = 1. \]

Can also show \[ \left[ G \right]_{x' = x} = 0. \]

There are "jump conditions" at \( x' = x \). \( \quad (10) \)

From (6), we know that

\[ G(x' > x) \propto \sinh[k(b - x')] \]

Since this \( \to 0 \) at \( x' = b \) \[ \text{[we have picked a good \textit{a.c.} \( e^{kx'} \text{and} -kx' \)]} \]
We also know that
\[ G(x' > x) \propto \cosh(kx'), \]
where, again, we have picked a good d.c. of \( e^{\pm kx'} \). For convenience, since we know that \( G(x' \to x) = G(x' \to x+ \to x) \), we may multiply the \( G \)'s on both sides by functions of \( x \) such that the continuity of \( G \) is manifest. In particular, we let
\[
\begin{align*}
G(x' \geq x) &= A \sinh[k(b-x')] \cosh[kx'], \\
G(x' < x) &= A \sinh[k(b-x')] \cosh[kx'].
\end{align*}
\]
where \( A = \text{const} \). These forms satisfy continuity at \( x' = x \).
We now apply the jump condition on the derivative, i.e., \[
\left[ \frac{dh}{dx} \right]_{x^+} = 1
\]

\[
\Rightarrow \quad -kA \cosh[k(b-x)] \cosh[kx] - kA \sinh[k(b-x)] \sinh[kx] = 1
\]

\[
\Rightarrow \quad A = \frac{-1}{k \cosh(kb)} \quad (12)
\]

\[
\Rightarrow \quad g(x) \text{ from (11) + (12)}
\]

In general, if one knows the homogeneous solutions \( h \), \( \mathcal{L} h = 0 \), \( h = \{h_1, h_2\} \), one can construct appropriate L.C.'s to write \( g \) on either side of \( x' = x \). Then, apply the jump conditions to fix the free constants.
To write the complete solution, note that we will need \( \frac{db}{dx'}\bigg|_{x'=b} \), to insert into the boundary term. From (10)

\[
\frac{db}{dx'}\bigg|_{x'=b} = \left. \frac{d}{dx'} \left[ A \sinh \left( k(b-x') \right) \cosh(kx') \right] \right|_{x'=b}
\]

\[= -Ak \cosh(kx) = \frac{\cosh(kx)}{\cosh(kb)} \]

Thus, the solution is

\[
y(x) = -\frac{\sinh[k(b-x)]}{k \cosh(kb)} \int_0^x dx' \cosh(kx') \sinh(kx') - \frac{\cosh(kx)}{k \cosh(kb)} \int_0^b dx' \sinh[k(b-x')] \sinh(kx') + \frac{\cosh(kx)}{\cosh(kb)}
\]

\[\Rightarrow y(b) = 1 \quad \text{(Typos?)}
\]

\[y'(0) = 0
\]