Important Properties
of Bessel Functions

(from Jackson, Classical E.D.)
The separation of variables is accomplished by the substitution:

\[ \Phi(\rho, \phi, z) = R(\rho)Q(\phi)Z(z) \]  
(3.72)

In the usual way this leads to the three ordinary differential equations:

\[ \frac{d^2 Z}{dz^2} - k^2 Z = 0 \]  
(3.73)

\[ \frac{d^2 Q}{d\phi^2} + \nu^2 Q = 0 \]  
(3.74)

\[ \frac{d^2 R}{d\rho^2} + \frac{1}{\rho} \frac{d R}{d\rho} + \left( k^2 - \frac{\nu^2}{\rho^2} \right) R = 0 \]  
(3.75)

The solutions of the first two equations are elementary:

\[ Z(z) = e^{zk} \]  
(3.76)

\[ Q(\phi) = e^{\pm \nu \phi} \]

For the potential to be single-valued when the full azimuth is allowed, \( \nu \) must be an integer. But barring some boundary-condition requirement in the \( z \) direction, the parameter \( k \) is arbitrary. For the present we assume that \( k \) is real and positive.

The radial equation can be put in a standard form by the change of variable \( x = \rho \). Then it becomes

\[ \frac{d^2 R}{dx^2} + \frac{1}{x} \frac{d R}{dx} + \left( 1 - \frac{\nu^2}{x^2} \right) R = 0 \]  
(3.77)

This is the Bessel equation, and the solutions are called Bessel functions of order \( \nu \). If a power series solution of the form

\[ R(x) = x^{\alpha} \sum_{j=0}^{\infty} a_j x^j \]  
(3.78)

is assumed, then it is found that

\[ \alpha = \pm \nu \]  
(3.79)

and

\[ a_{2j} = -\frac{1}{4j(j + \alpha)} a_{2j-2} \]  
(3.80)
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for \( j = 1, 2, 3, \ldots \). All odd powers of \( x^j \) have vanishing coefficients. The recursion formula can be iterated to obtain

\[
a_{2j} = \frac{(-1)^j \Gamma(\alpha + 1)}{2^{2j+1} \Gamma(j + \alpha + 1)} a_0 \quad (3.81)
\]

It is conventional to choose the constant \( a_0 = [2^n \Gamma(\alpha + 1)]^{-1} \). Then the two solutions are

\[
J_\nu(x) = \left(\frac{x}{2}\right)^\nu \sum_{j=0}^{\infty} \frac{(-1)^j}{\Gamma(j + \nu + 1)} \left(\frac{x}{2}\right)^{2j} \quad (3.82)
\]

\[
J_{-\nu}(x) = \left(\frac{x}{2}\right)^{-\nu} \sum_{j=0}^{\infty} \frac{(-1)^j}{\Gamma(j - \nu + 1)} \left(\frac{x}{2}\right)^{2j} \quad (3.83)
\]

These solutions are called Bessel functions of the first kind of order \( \pm \nu \). The series converge for all finite values of \( x \). If \( \nu \) is not an integer, these two solutions \( J_{\pm\nu}(x) \) form a pair of linearly independent solutions to the second-order Bessel equation. However, if \( \nu \) is an integer, it is well known that the solutions are linearly dependent. In fact, for \( \nu = m \), an integer, it can be seen from the series representation that

\[
J_{-m}(x) = (-1)^m J_m(x) \quad (3.84)
\]

Consequently it is necessary to find another linearly independent solution when \( \nu \) is an integer. It is customary, even if \( \nu \) is not an integer, to replace the pair \( J_{\pm\nu}(x) \) by \( J_\nu(x) \) and \( N_\nu(x) \), the Neumann function (or Bessel function of the second kind):

\[
N_\nu(x) = \frac{J_\nu(x) \cos \nu \pi - J_{-\nu}(x)}{\sin \nu \pi} \quad (3.85)
\]

For \( \nu \) not an integer, \( N_\nu(x) \) is clearly linearly independent of \( J_\nu(x) \). In the limit \( \nu \to \) integer, it can be shown that \( N_\nu(x) \) is still linearly independent of \( J_\nu(x) \). As expected, it involves \( \log x \). Its series representation is given in the reference books.

The Bessel functions of the third kind, called Hankel functions, are defined as linear combinations of \( J_\nu(x) \) and \( N_\nu(x) \):

\[
H^{(1)}_\nu(x) = J_\nu(x) + iN_\nu(x) \\
H^{(2)}_\nu(x) = J_\nu(x) - iN_\nu(x) \quad (3.86)
\]

The Hankel functions form a fundamental set of solutions to the Bessel equation, just as do \( J_\nu(x) \) and \( N_\nu(x) \).

The functions \( J_\nu, N_\nu, H^{(1)}_\nu, H^{(2)}_\nu \) all satisfy the recursion formulas

\[
\Omega_{\nu-1}(x) + \Omega_{\nu+1}(x) = \frac{2\nu}{x} \Omega_{\nu}(x) \quad (3.87)
\]

\[
\Omega_{\nu-1}(x) - \Omega_{\nu+1}(x) = 2 \frac{d\Omega_{\nu}(x)}{dx} \quad (3.88)
\]

where \( \Omega_{\nu}(x) \) is any one of the cylinder functions of order \( \nu \). These may be verified directly from the series representation (3.82).
For reference purposes, the limiting forms of the various kinds of Bessel function are given for small and large values of their argument. For simplicity, we show only the leading terms:

\[ x \ll 1 \quad J_\nu(x) \to \frac{1}{\Gamma(\nu + 1)} \left( \frac{x}{2} \right)^\nu \]
\[ x \to 0 \]  
**asymptotic behavior**

\[ N_\nu(x) \to \begin{cases} \\ \
\frac{2}{\pi} \left( \ln \left( \frac{x}{2} \right) + 0.5772 \cdots \right) & , \quad \nu = 0 \\
-\frac{\Gamma(\nu)}{\pi} \left( \frac{2}{x} \right)^\nu & , \quad \nu \neq 0 \\
\end{cases} \]

In these formulas \( \nu \) is assumed to be real and nonnegative.

\[ x \gg 1, \nu \quad J_\nu(x) \to \sqrt{\frac{2}{\pi x}} \cos \left( x - \frac{\nu \pi}{2} - \frac{\pi}{4} \right) \]
\[ N_\nu(x) \to \sqrt{\frac{2}{\pi x}} \sin \left( x - \frac{\nu \pi}{2} - \frac{\pi}{4} \right) \]

The transition from the small \( x \) behavior to the large \( x \) asymptotic form occurs in the region of \( x \sim \nu \).

From the asymptotic forms (3.91) it is clear that each Bessel function has an infinite number of roots. We will be chiefly concerned with the roots of \( J_\nu(x) \):

\[ I_\nu(x_m) = 0 \quad (n = 1, 2, 3, \ldots) \]

\( x_m \) is the \( n \)th root of \( J_\nu(x) \). For the first few integer values of \( \nu \), the first three roots are:

\[ \nu = 0, \quad x_{00} = 2.405, 5.520, 8.654, \ldots \]
\[ \nu = 1, \quad x_{10} = 3.832, 7.016, 10.173, \ldots \]
\[ \nu = 2, \quad x_{20} = 5.136, 8.417, 11.620, \ldots \]

For higher roots, the asymptotic formula

\[ x_{nm} = n\pi + \left( \nu - \frac{1}{2} \right) \frac{\pi}{2} \]

gives adequate accuracy (to at least three figures). Tables of roots are given in *Jahnke, Emde, and Lösch* (p. 194) and *Abramowitz and Stegun* (p. 409).

Having found the solution of the radial part of the Laplace equation in terms of Bessel functions, we can now ask in what sense the Bessel functions form an orthogonal, complete set of functions. We consider only Bessel functions of the first kind, and we show that \( \sqrt{\rho} J_\nu(x_m \rho/a) \), for fixed \( \nu \geq 0, n = 1, 2, \ldots \), form an orthogonal set on the interval \( 0 \leq \rho \leq a \). The demonstration starts with the differential equation satisfied by \( J_\nu(x_m \rho/a) \):

\[ \frac{1}{\rho} \frac{d}{d\rho} \left[ \rho \frac{d}{d\rho} \left( \frac{J_\nu(x_m \rho/a)}{\rho} \right) \right] + \left( \frac{x_m^2}{a^2} - \frac{\nu^2}{\rho^2} \right) J_\nu(x_m \rho/a) = 0 \]

(3.93)
If we multiply the equation by \( \rho J_\nu(x_m \rho/a) \) and integrate from 0 to \( a \), we obtain

\[
\int_0^a J_\nu(x_m \rho/a) \frac{d}{d\rho} \left[ \rho \frac{dJ_\nu(x_m \rho/a)}{d\rho} \right] d\rho = - \int_0^a \rho \frac{dJ_\nu(x_m \rho/a)}{d\rho} \frac{dJ_\nu(x_m \rho/a)}{d\rho} d\rho + \int_0^a \left( \frac{x_m^2 \rho^2}{a^2} - \frac{\nu^2}{\rho^2} \right) \rho J_\nu(x_m \rho/a) J_\nu(x_m \rho/a) d\rho = 0
\]

Integration by parts, combined with the vanishing of \( \rho J_\nu(x) \) at \( \rho = 0 \) (for \( \nu \geq 0 \)) and \( \rho = a \), leads to the result:

\[
\int_0^a \rho \frac{dJ_\nu(x_m \rho/a)}{d\rho} \frac{dJ_\nu(x_m \rho/a)}{d\rho} d\rho + \int_0^a \left( \frac{x_m^2 \rho^2}{a^2} - \frac{\nu^2}{\rho^2} \right) \rho J_\nu(x_m \rho/a) J_\nu(x_m \rho/a) d\rho = 0
\]

If we now write down the same expression, with \( n \) and \( n' \) interchanged, and subtract, we obtain the orthogonality condition:

\[
(x_m^2 - x_m^{2'}) \int_0^a \rho J_\nu(x_m \rho/a) J_\nu(x_m \rho/a) d\rho = 0 \quad (3.94)
\]

Adroit use of the differential equation, and the recursion formulas (3.87) and (3.88) leads to the normalization integral:

\[
\text{Orthogonality and normalization} \quad \int_0^a \rho J_\nu(x_m \rho/a) J_\nu(x_m \rho/a) d\rho = \frac{a^2}{2} [J_{\nu+1}(x_m)]^2 \delta_{n,n'} \quad \text{no \( k \), \( \lambda \) labels}
\]

Assuming that the set of Bessel functions is complete, we can expand an arbitrary function of \( \rho \) on the interval \( 0 \leq \rho \leq a \) in a Fourier–Bessel series:

\[
f(\rho) = \sum_{n=1}^\infty A_n J_\nu(x_m \rho/a) \quad \text{on } [0,a] \quad (3.96)
\]

where

\[
A_n = \frac{2}{a^2 J_{\nu+1}(x_m)} \int_0^a \rho f(\rho) J_\nu(x_m \rho/a) d\rho \quad (3.97)
\]

Our derivation of (3.96) involved the restriction \( \nu \geq 0 \). Actually it can be proved to hold for all \( \nu \geq -1 \).

Expansion (3.96) and (3.97) is the conventional Fourier–Bessel series and is particularly appropriate to functions that vanish at \( \rho = a \) (e.g., homogeneous Dirichlet boundary conditions on a cylinder; see the following section). But it will be noted that an alternative expansion is possible in a series of functions \( \sqrt{\rho} J_\nu(y_m \rho/a) \) where \( y_m \) is the \( m \)-th root of the equation \( dJ_\nu(x) \)/\( dx = 0 \). The reason is that, in proving the orthogonality of the functions, all that is demanded is that the quantity \( \rho J_\nu(\lambda \rho)(dJ_\nu/d\rho)J_\nu(k\rho) - \rho J_\nu(k\rho)(dJ_\nu/d\rho)J_\nu(\lambda \rho) \) vanish at the end points \( \rho = 0 \) and \( \rho = a \). The requirement is met by \( \lambda = y_m/a \) or \( \lambda = y_m/a \), where \( J_\nu(x_m) = 0 \) and \( J'_\nu(y_m) = 0 \), or, more generally, by \( \rho (dJ_\nu/d\rho)J_\nu(k\rho) + J_\nu(k\rho) = 0 \) at the end points, with \( \lambda \) a constant independent of \( k \). The expansion in terms of the set \( \sqrt{\rho} J_\nu(y_m \rho/a) \) is especially useful for functions with vanishing slope at \( \rho = a \). (See Problem 3.11.)
A Fourier–Bessel series is only one type of expansion involving Bessel functions. Some of the other possibilities are:

- Neumann series: $\sum_{n=0}^{\infty} a_n J_{\nu+n}(z)$
- Kapteyn series: $\sum_{n=0}^{\infty} a_n J_{\nu+n}((\nu+n)z)$
- Schlömilch series: $\sum_{n=1}^{\infty} a_n I_n(nx)$

The reader may refer to Watson (Chapters XVI–XIX) for a detailed discussion of the properties of these series. Kapteyn series occur in the discussion of the Kepler motion of planets and of radiation by rapidly moving charges (see Problems 14.14 and 14.15).

Before leaving the properties of Bessel functions, we note that if, in the separation of the Laplace equation, the separation constant $k^2$ in (3.73) had been taken as $-k^2$, then $Z(z)$ would have been $\sin kx$ or $\cos kx$ and the equation for $R(\rho)$ would have been:

$$\frac{d^2R}{d\rho^2} + \frac{1}{\rho} \frac{dR}{d\rho} - \left( k^2 + \frac{\rho^2}{\rho^2} \right) R = 0 \quad (3.98)$$

With $k\rho = x$, this becomes

$$\frac{d^2R}{dx^2} + \frac{1}{x} \frac{dR}{dx} - \left( 1 + \frac{x^2}{x^2} \right) R = 0 \quad (3.99)$$

The solutions of this equation are called modified Bessel functions. It is evident that they are just Bessel functions of pure imaginary argument. The usual choices of linearly independent solutions are denoted by $I_\nu(x)$ and $K_\nu(x)$. They are defined by

$$I_\nu(x) = i^{-\nu}J_\nu(ix) \quad (3.100)$$
$$K_\nu(x) = \frac{\pi}{2} i^{\nu+1}H^{(1)}_{\nu}(ix) \quad (3.101)$$

and are real functions for real $x$ and $\nu$. Their limiting forms for small and large $x$ are, assuming real $\nu \geq 0$:

$$x \ll 1 \quad I_\nu(x) \to \frac{1}{\Gamma(\nu + 1)} \left( \frac{x}{2} \right)^\nu \quad (3.102)$$

$$K_\nu(x) \to \begin{cases} -\left[ \ln \left( \frac{x}{2} \right) + 0.5772 \cdots \right], & \nu = 0 \\ \frac{\Gamma(\nu)}{2} \left( \frac{2}{x} \right)^\nu, & \nu \neq 0 \end{cases} \quad (3.103)$$

$$x \gg 1, \nu \quad I_\nu(x) \to \frac{1}{\sqrt{2\pi x}} e^x \left[ 1 + 0 \left( \frac{1}{x} \right) \right] \quad (3.104)$$

$$K_\nu(x) \to \sqrt{\frac{\pi x}{2}} e^{-x} \left[ 1 + 0 \left( \frac{1}{x} \right) \right]$$
(b) Show that the total current flowing out through the upper hemisphere of the sphere is

\[ I = \frac{2\sigma' \sigma}{\sigma + 2\sigma'} \cdot m^2 F \]

Calculate the total power dissipation outside the sphere. Using the lumped circuit relations, \( P = I^2 R_t = IV_e \), find the effective external resistance \( R_e \) and voltage \( V_e \).

(c) Find the power dissipated within the sphere and deduce the effective internal resistance \( R_i \) and voltage \( V_i \).

(d) Define the total voltage through the relation, \( V_i = (R_e + R_i)I \) and show that \( V_i = 4\pi F L^3 \), as well as \( V_e + V_i = V_r \). Show that \( IV_i \) is the power supplied by the “chemical” force.


3.16 (a) Starting from the Bessel differential equation and appropriate limiting procedures, verify the generalization of (3.108),

\[ \frac{1}{k} \delta(k - k') = \int_0^\infty \rho J_\nu(k\rho) J_\nu(k'\rho) \, d\rho \]

or equivalently that

\[ \frac{1}{\rho} \delta(\rho - \rho') = \int_0^\infty k J_\nu(k\rho) J_\nu(k'\rho) \, dk \]

where \( \text{Re}(\nu) > -1 \).

(b) Obtain the following expansion:

\[ \frac{1}{|x - x'|} = \sum_{m=-n}^n \int_0^\infty \frac{k}{k^2 + z^2} e^{ikz} e^{i\nu \rho} J_m(k\rho) J_m(k\rho') e^{-ik|z|} \, dk \]

(c) By appropriate limiting procedures prove the following expansions:

\[ \frac{1}{\sqrt{\rho^2 + z^2}} = \int_0^\infty e^{-kz} J_0(k\rho) \, dk \]

\[ J_\nu(k\sqrt{\rho^2 + z^2} - 2\rho\rho' \cos \phi) = \sum_{m=-n}^n e^{i\mu \phi} J_m(k\rho) J_m(k\rho') \]

\[ e^{i\mu \phi} = \sum_{m=-n}^n e^{i\mu \phi} J_m(k\rho) \]

(d) From the last result obtain an integral representation of the Bessel function:

\[ J_m(x) = \frac{1}{2i\pi} \int_0^{2\pi} e^{i\mu \phi - im \phi} \, d\phi \]

Compare the standard integral representations.

3.17 The Dirichlet Green function for the unbounded space between the planes at \( z = 0 \) and \( z = L \) allows discussion of a point charge or a distribution of charge between parallel conducting planes held at zero potential.

(a) Using cylindrical coordinates show that one form of the Green function is

\[ G(x, x') = \frac{4}{L} \sum_{n=-\infty}^{\infty} \sum_{m=-\infty}^{\infty} e^{in\phi - \phi'} \sin \left( \frac{n\pi z}{L} \right) \sin \left( \frac{n\pi z'}{L} \right) J_m \left( \frac{n\pi}{L} \rho \right) K_m \left( \frac{n\pi}{L} \rho' \right) \]
Some Bessel Funcs

BESSEL FUNCTIONS OF INTEGER ORDER

Figure 9.1. $J_0(x), Y_0(x), J_1(x), Y_1(x)$.

Figure 9.2. $J_{10}(x), Y_{10}(x)$, and $M_{10}(x) = \sqrt{J_{10}(x) + Y_{10}(x)}$.

Figure 9.3. $J_{10}(x)$ and $Y_{10}(x)$.

Bessel($z$), complex space.

Note branch cut

Figure 9.4. Contour lines of the modulus and phase of the Hankel Function $H_0^{(2)}(x + iy) = M_{0, y}^2$. From E. Jahnke, F. Emde, and F. Lösch, Tables of higher functions, McGraw-Hill Book Co., Inc., New York, N.Y., 1960 (with permission).