High Energy States for Quantum Harmonic Oscillator, by WKB method

\[ H = \frac{p^2}{2m} + \frac{1}{2} mw_0^2 x^2 \]

\[ \Rightarrow -\frac{\hbar^2}{2m} \Psi_{xx} = E - \frac{1}{2} mw_0^2 x^2 \]

\[ -\frac{\hbar^2}{m^2 w_0^2 a^2} \Psi_{xx} = \frac{E}{\frac{1}{2} mw_0^2 a^2} - \frac{x^2}{a^2}, \text{ } a^2 \text{ to be defined} \]

Let \( a^2 \) be defined as \[ \frac{\hbar^2}{m^2 w_0^2 a^4} = 1 \]

Let \( \frac{x}{a} \to x \), \( \frac{E}{\frac{1}{2} mw_0^2 a^2} \to E \)

\[ \Psi_{xx} = -(E - x^2) \Psi \]

Assume \( E \gg 1 \)

\( \Psi_{xx} \) is "small"

\( \Rightarrow \) use WKB ansatz
Try \( \Psi = e^S \) \( |S| \gg 1 \) ansatz

\[ \Rightarrow \Psi_{xx} = (S'^2 + S'') \Psi \]

\[ \Rightarrow S'^2 + S'' = - (E - x^2) \]

let \( k(x) = (E - x^2) \)

\[ \Rightarrow S'^2 + S'' = - k^2 \]

**Lowest order** \( S_0'^2 = - k^2 \) \( S_0 = \pm ik(x) \)

\[ S_0 = \int \lambda x \ dx \ k(x) \]

**1st order** \( 2S_0 S_1' + S_0'' = 0 \)

\[ \Rightarrow S_1' = - \frac{S_0''}{2S_0} \Rightarrow S_1 = - \frac{k}{2k} \]

\[ \Rightarrow \Psi_0 = e^{+ ikx} \]

Assume symmetry \( \Psi_0 (x) = \pm \Psi_0 (x) \)

\[ \Psi_0 (x) = \left\{ \begin{array}{ll}
\cos \left[ S_0 \int \lambda x' \ k(x') \right] & \text{even for } |x| < \sqrt{E} \\
\sin \left[ S_0 \int \lambda x' \ k(x') \right] & \text{odd} \end{array} \right. \]
Note, for $|x| > \sqrt{E}$, solutions are evanescent. Boundary conditions are $\Psi (|x| \to \infty) \to 0$.

$$y_0 (x) = \mathcal{C} e^{-\int k dx}$$

$$k = \sqrt{x^2 - E}$$

Self consistency check

Need $|\xi_1| \ll |s_0|$ . Try $|\xi_1| \ll |s_0|$

$$\Rightarrow \left| \frac{k_1}{k} \right| \ll k, \quad k_1 \ll k^2$$

$$k^2 = E - x^2, \quad 2 k k_1 = -2 x$$

$$\Rightarrow x \ll k^3 \Rightarrow x^{2/3} \ll E - x^2$$

$x$ ranges up to $-E^{1/2}$, $E^{1/3} \ll E - x^2$

Clearly, there is a failure at the turning point $x \to \sqrt{E}$. The condition for validity is the

$$E^{1/3} \ll (\sqrt{E} - x) 2 \sqrt{E}$$

$$\frac{1}{E^{1/6}} \ll |E^{1/2} - x| \quad \text{as} \quad x \to E^{1/2}$$
The WKB solutions 1 + 2 are good only for \( |\Delta x| \gg E^{-1/6} \).

For \( \Delta x \sim \sqrt{E} V / 6 \), we must examine the SE near \( x \to E^{1/2} \).

Consider SE as \( x \to E^{1/2} \)

\[ \Rightarrow \psi_{xx} = -(\sqrt{E} - x)(\sqrt{E} + x) \psi \]

\[ \Rightarrow \psi_{xx} = -(\sqrt{E} - x) 2 \sqrt{E} \psi \]

let \[ (x - \sqrt{E}) = \frac{t}{2^{1/3} E^{1/6}} \] (4a)

\[ \Rightarrow \psi_{tt} = t \psi \] "Layer equation" (3)

Valid for \( t \ll 1 \)

\[ \Rightarrow (x - \sqrt{E}) \sim \sqrt{E} V / 6 \]

Solutions for 3 are \( A(t), B(t) \)

But we want \( \psi(t \to \infty) \to 0 \)

\[ \Rightarrow \psi(t) = D A(t) \]

\( D = \text{const} \) (4)
We must now do an asymptotic matching. This means

\[
\Psi_{\text{outer}} (x \to \pm E) \leftrightarrow \Psi_{\text{layer}} (|t| \gg 1, t \ll 0) \]

\[
\Psi_{\text{outer}} (x \to \mp E) \leftrightarrow \Psi_{\text{layer}} (|t| \gg 1, t > 0) \]

\( \leftrightarrow \) means the LTHS & RTHS should be identical term by term & for all orders.

From our integral solution to \( \text{Ai}(t) \), we know \( \text{Ai}(|t| \gg 1) \) asymptotically. We found

\[
\text{Ai}(t \to +\infty) \to \frac{1}{2\sqrt{\pi}} \frac{e^{-\frac{2}{3}t^{3/2}}}{t^{1/4}}
\]

\[
\text{Ai}(t \to -\infty) \to \frac{\cos \left[ \frac{\pi}{4} - \frac{2}{3} (-t)^{3/2} \right]}{\sqrt{\pi} (-t)^{1/4}}
\]
We also need \( Y_{\text{outer}} (x \to \sqrt{E}) \).

Consider \( \int_{0}^{\sqrt{E}} k \, dx' \), \( x \to \sqrt{E} \)

\[
\int_{0}^{\sqrt{E}} k \, dx' \to \int_{0}^{\sqrt{E}} k \, dx - \int_{x}^{\sqrt{E}} k \, dx',
\]

\[
k^2 = E - x^2 \to (\sqrt{E} - x) 2\sqrt{E}
\]

\[
\Rightarrow \int_{0}^{\sqrt{E}} k \, dx' \to \int_{0}^{\sqrt{E}} k \, dx - \int_{x}^{\sqrt{E}} k \, dx' (\sqrt{E} - x)^{1/2} 2^{1/2} \sqrt{E}^{1/4}
\]

\[
\to \int_{0}^{\sqrt{E}} k \, dx - \frac{2}{3} (\sqrt{E} - x)^{3/2} 2^{1/2} E^{1/4}
\]

Now match in the region \( x \to \sqrt{E} \) from left:

\[
\Rightarrow \cos \left[ \int_{0}^{\sqrt{E}} k \, dx - \frac{2}{3} (\sqrt{E} - x)^{3/2} 2^{1/2} E^{1/4} \right] \longleftrightarrow
\]

\[
\frac{D}{\sqrt{\pi}} \frac{\cos \left[ \frac{\pi}{4} - \frac{2}{3} (\sqrt{E} - x)^{3/2} \right]}{(-t)^{1/4}} \]

\[
= \frac{D}{\sqrt{\pi}} \frac{1}{(-t)^{1/4}} \cos \left[ \frac{\pi}{4} - \frac{2}{3} (\sqrt{E} - x)^{3/2} 2^{1/2} \sqrt{E}^{1/4} \right]
\]

using (4a). Note the matching. The \( D \) can also be determined (also need to keep \( t^{1/2} \) terms in the cosine). What remains is the remaining phase.
The phase matching yields

\[ \int_{0}^{\frac{\sqrt{E}}{k}} k \, dx = \frac{x}{2} + n\pi, \quad n = 0, 1, 2, \ldots \]

This is the Bohr-Sommerfeld quantization condition
(for the even modes; odd modes can be done likewise)

\[ \Rightarrow \quad \int_{0}^{1} \int_{0}^{1} (1-s^2)^{1/2} \, ds \, (n+\frac{1}{4}) \pi = \text{energy eigenvalue} \]

We had assumed \( E \gg 1 \). This is only self-consistent if \( n \gg 1 \).