Discussion on standard perturbation methods in relation to scaling methods used to analyze asymptotic behavior of ODE's near singular points.

4 examples are given to illustrate specific methods and interrelationship.
Example for "standard perturbation theory"

0. Suppose \( y'' + y = 2 \varepsilon y \sin x \)

You are given that \( \varepsilon \ll 1 \),
and asked to solve this by perturbation theory. Standard approach is

0. Let \( y = \sum_{n=0}^{\infty} \varepsilon^n y_n \), plug in, and

equate \( \varepsilon \) orders, i.e.,

\[
\sum_{n=0}^{\infty} \varepsilon^n (y_n'' + y_n) = 2 \varepsilon \sin x \sum_{n=0}^{\infty} \varepsilon^n y_n
\]

\(\varepsilon^0 \Rightarrow \)

\( y_0'' + y_0 = 0 \)

\(\varepsilon^1 \Rightarrow \)

\( y_1'' + y_1 = 2 \sin x y_0 \), ... etc

\(\Rightarrow \) \( y_0 = \left\{ \begin{array}{ll}
\frac{\sin x}{\cos x} & \text{if } \varepsilon \ll 1 \\
\frac{1 - \cos 2x}{\sin 2x} & 
\end{array} \right. \)

\( y_1'' + y_1 = \left\{ \begin{array}{ll}
\frac{1 + \frac{1}{3} \cos 2x}{\sin 2x} & \text{Need to ensure small terms}
\end{array} \right. \)

\(\Rightarrow \)

\( y = \left\{ \begin{array}{ll}
\frac{\sin x}{\cos x} & \text{if } \varepsilon \ll 1 \\
\frac{1 + \frac{1}{3} \cos 2x}{\sin 2x} & 
\end{array} \right. + \ldots...

\)

0. Note: This method makes no assumption on the range or domain of \( x \).
Asymptotic behavior of Airy function

as \( x \to 0 \): Example of finding how a solution behaves in some small domain in \( x \) using scaling methods

Airy eqn: \[ y'' = xy \]

Find \( y \) as \( x \to 0 \) perturbatively

Method:
1. Scale the equation \( \Rightarrow \frac{1}{x}; x = \delta x^3 \)
2. Ansatz as \( x \to 0 \), expect \( \text{RHS} \to 0 \), assume \( \text{perturbatively} \)
   \( \Rightarrow \text{expand assuming RHS is small} \)
3. Rigor check to make sure \( |y_{n+1}| \ll |y_n| \)

Let \( y = y_0 + y_1 + y_2 \)

\((y_0 + y_1 + \ldots)'' = \frac{x}{x_0 + y_1 + \ldots} \)

"small"

\[ \begin{align*}
\text{lowest order} & \quad y_0'' = 0 \quad \Rightarrow \quad y_0 = \left\{ \frac{1}{3} x^3 \right\} \\
\text{1st order} & \quad y_1'' = xy_0 \quad \Rightarrow \quad y_1 = \left\{ \frac{x^3}{3} \right\} \\
& \quad \Rightarrow \quad y = \left\{ \frac{x^3}{3} \right\} + \ldots + \left\{ \frac{x^4}{4 \cdot 3} \right\} \\
& \quad \Rightarrow \quad y = \left\{ \frac{1 + x^3}{3 \cdot 2} \right\} \ldots + x + \frac{x^4}{4 \cdot 3} \ldots \}
\end{align*} \]

Self consistent check: \( |y| \ll |y_0| \Rightarrow x \ll 1 \), so good approach.
Asymptotic behavior of Airy as $x \to \infty$.

Same as previous but recast as

"Standard perturbation theory."

\[ y'' = xy \]

Find $y(x)$ as $x \to \infty$.

Standard method: since $x$ is small,

redefine (rescale) $x \equiv \xi$, where $\xi \ll 1$,

\[ \Rightarrow y_{ss} = \xi^3 y \]

rescaled equation

Now solve this as in (P1) by pert theory

as $\xi \to 0$. Let $y = \sum_{n=0}^{\infty} y_n(\xi)$

\[ \Rightarrow \sum_{n=0}^{\infty} \frac{d^2 y_n}{d \xi^2} = \xi^{n+3} y_n \]

$\xi^0$: $y_{0ss} = 0 \Rightarrow y_0 = \xi \Rightarrow y$

$\xi^1$: $y_{1ss} = 0 \Rightarrow y_1$ same as $y_0 \Rightarrow$ can set to zero

$\xi^2$: $y_{2ss} = 0 \Rightarrow y_2 = 0$

$\xi^3$: $y_{3ss} = \xi y_0 \Rightarrow y_3 = \xi \left( \frac{\xi^3}{3 \cdot 2} \right) y_0 \Rightarrow y_3 = x/y$

\[ \Rightarrow y = \left[ 1 + \frac{x^3}{3 \cdot 2} + \cdots \right] \frac{1}{\xi} \left( x + \frac{x^4}{4 \cdot 3} + \cdots \right)^2 \]

Since the overall constant does not matter, this solution is identical to the "scaling method" one on (P2).
An equation that illustrates how "standard perturbation theory" could yield insufficient information.

Suppose given that \( \epsilon y'' + y' + y = 0 \)

and \( \epsilon \ll 1 \).

Suppose asked to solve by standard perturbation theory. Proceeding as usual,

\[
\sum_{n=0}^{\infty} \left[ \epsilon^{n+1} y_n'' + \epsilon^n y_n' + \epsilon^n y_n \right] = 0
\]

\( \epsilon^0 \Rightarrow y_0'' + y_0 = 0 \Rightarrow y_0 = e^{-x} \)

\( \epsilon^1 \Rightarrow y_1'' + y_1 + y_0'' = 0 \Rightarrow y_1'' + y_1 = -e^{-x}, \ y_1 = -xe^{-x} \)

\( \Rightarrow \left\{ y = e^{-x} - ex e^{-x} + \ldots \right\} \ \epsilon \ll 1 \)

This is a good series as \( \epsilon \to 0 \).

But, note, we only found one solution for a 2nd order ODE. We "lost" the other solution.

Remedy: use the WKB ansatz \( y = e^S, |S| > 1 \)

This is an example of "singular perturbation theory."