Example for Reduction of Order

- Airy functions

Consider \( y'' = xy \)  \( \Box \)  

highest power \( y \times \) in coeffs

\(< \) highest order of \( d/dx \)

\( = ) \) advantageous to try generalized integral tran form.

Try \( y(x) = \int_c dk \, e^{kx} f(k) \)  \( \text{(general form)} \)

Contour is in \( k \)-space, from \( 1 \rightarrow 2 \).
Assume end points and \( C \) are independent of \( x \).

\( = ) \) \( \frac{dy}{dx} = \int_c dk \, e^{kx} k f \)

\( \frac{d^2y}{dx^2} = \int_c dk \, e^{kx} k^2 f \)
also, \( xy = \int dk \ x e^{kx} f(k) \)
\[
= \sum \int dk \ (\frac{d}{dk} e^{kx}) f(k) = \left[ e^{kx} f(k) \right]_1^0 \]
\[
- \int dk \ e^{kx} \frac{df}{dk} \]

\* Assume that end point contributions vanish \( \left[ e^{kx} f(k) \right]_1^0 = 0 \),
to be verified after we find \( f \), etc.

Then, \( y'' = xy \) becomes
\[
\int dk \ e^{kx} \left[ k^2 f + df/dk \right] = 0 \]
\[
\sum \int dk \ e^{kx} \left[ k^2 f + \frac{df}{dk} \right] = 0 \]

\[ \frac{df}{dk} = -k^2 f \]  (2)

we have reduced 2nd order eqn (1)
to 1st order eqn (2)
\[ \Rightarrow \text{reduction of order} \]
\[ \Rightarrow \text{useful method} \]
\[ (2) \implies f(k) = Ce^{-\frac{k^3}{3}} \]

But we need to satisfy \( [e^{ikx}f(k)]^2 = 0 \)

\[ \text{i.e. } e^{ikx}e^{-\frac{k^3}{3}} \to 0 \text{ in } k \text{ space} \]

Here, \( \text{Re}k \to +\infty \) is one possibility.

In general, as \( |k| \to \infty \), we try to see if \( e^{-\frac{k^3}{3}} \to 0 \). Let \( k = r e^{i\theta} \)

\[ \implies \text{end points vanish for } \text{Re}(k^3) > 0 \]

\[ \implies \cos(3\theta) > 0 \implies -\frac{\pi}{2} < 3\theta < \frac{\pi}{2} \]

On \( \frac{3\pi}{2} < 3\theta < \frac{5\pi}{2} \), \( -\frac{\pi}{2} < 3\theta < -\frac{3\pi}{2} \)

\[ \implies -\frac{\pi}{6} < \theta < \pi/6, \quad \pi/2 < \theta < 5\pi/6, -5\pi/6 < \theta < -\pi/6 \]

Send points vanish
Thus, there are 3 possible contours: $C_+$, $C_-$, and $C$ except $C = C_+ + C_-$ are related. 

Thus, there are two solutions:

\[
Y_+ (x) = \int\left. dk \right|_{C_+} e^{kx} e^{-h^2/3} \]
\[
Y_A (x) = \int\left. dk \right|_{C_A} e^{kx} e^{-h^2/3} \]

2 Linearly independent solutions. Given as integral representations.

These are the Airy functions, $Ai(x)$, $Bi(x)$.

Reduction of order works as above. In general, though building independent contours may involve different maneuvers.