

Summary of Theorems (Ch 1, Sheng)

Theorems	Description
1. $L\{af(t) + bg(t)\} = aL\{f(t)\} + bL\{g(t)\}$ Corrolat 1a.) For any piecewise continuous f : $\lim_{s \rightarrow \infty} L\{f(t)\} = 0$	Linearity and asymptotic property of LT
2. $L\{D_t f(t)\} = sL\{f(t)\} - f(0)$ 2a. Corrolary to 2 (not found in Sheng): $L\{D_t^n f(t)\} = s^n L\{f(t)\} - \sum_{j=1}^n s^{n-j} f^{(j-1)}(0)$ 4 : $L\{D_t^{-1} f(t)\} = 1/s L\{f(t)\}$	2. & 4. give formulae in terms of the LT of f (i.e., $L\{f(t)\} = F(s)$) when you're faced with computing the LT for the derivative of f (or in the case of Thm 2a, the n -th derivative) or the antiderivative of f ($D_t^{-1}\{f(t)\} \equiv \int_0^t f(\omega) d\omega$)
3. $L\{f(t)\} = F(s) \Rightarrow L\{f(at)\} = 1/a F(s/a)$	An obviously useful simplification formula, but it also tells you that the LT is <i>not</i> linear with respect to the <i>variable</i> arguments of f . LT is linear with respect to <i>its functional</i> arguments ¹ !
5. $L\{t^n f(t)\} = (-1)^n D_s^n L\{f(t)\}$ for $n = 0, 1, 2, \dots$	Again, another obviously useful simplification formula
5a. Corrolary (Example 1-2-4) $L\{t^p\} = \frac{\Gamma(p+1)}{s^{p+1}}$, for any $p > -1$	(From Handout 1c, Formula 1c.1) The formula comes especially in handy of integer and half-integer exponents, given Properties 1-4 as of the Gamma function as summarized in Handout 1c
6. $L\{t^{-1} f(t)\} = \int_s^\infty F(\omega) d\omega$ where: $L\{f(t)\} = F(s)$	Discussed here (Handout 2). Calculus analogy: Recall the power-rule for integration did <i>not</i> hold for t^p when $p = -1$? (You had to introduce a rigorous defn of logarithm)
7. $L\{f(t)\} = F(s) \Rightarrow L\{e^{at} f(t)\} = F(s - a)$	Shifting Thm 1 (discussed here)
8. $L\{f(t)\} = F(s) \Rightarrow L\{u(t-a)f(t-a)\} = e^{-as} F(s)$	Shifting Thm2 (discussed here)
$f(t+p) = f(t) \Rightarrow$ 9. $L\{f(t)\} = \frac{1}{1 - e^{-sp}} \int_0^p f(t) e^{-st} dt$	The periodic function theorem, it's 'simplifying' insofar as the LT gets reduced to a finite, instead of an improper integral (good news ☺) Bad news: you usually have to evaluate the thing by hand
10.) & 11.) : $\lim_{(t \rightarrow 0^+) \vee (t \rightarrow \infty)} f(t) = \lim_{(s \rightarrow \infty) \vee (s \rightarrow 0)} \frac{1}{s} F(s)$	Relates the initial and asymptotic behavior of a function with its associated LT
12.)	The Convolution Theorem. Enables one to simplify determining an LT of a function h and its inverse LT of H if one knows the LTs of of

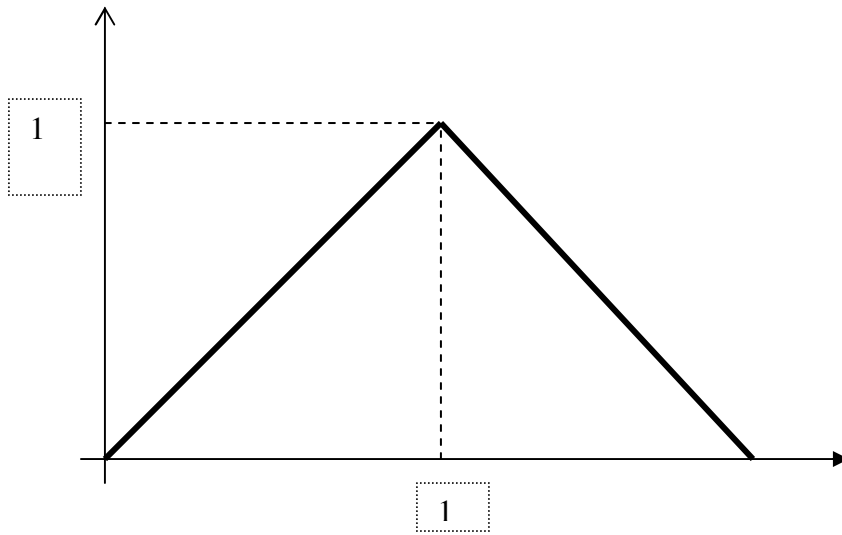
¹ I.e., as Thm1 states: $L\{af(t) + bg(t)\} = aL\{f(t)\} + bL\{g(t)\}$.

However: $L\{f(at + bt)\} \neq aL\{f(t)\} + bL\{f(t)\}$. This tells us LT is a linear operator (it's linear with respect to functions, which are its arguments), but it is certainly *not* a linear function (it's not linear with respect to the function's variable arguments.)

$F(s)G(s) = L\{(f * g)(t)\},$ $(f * g)(t) = \int_0^t f(u)g(t-u)du = \int_0^t f(t-u)g(u)du$	H 's 'factors' (I.e.: if one can write: $H(s) = F(s)G(s)$)
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Some (selected) examples (ch1)

- **Problem 4, p 41 (Sheng)**



$$f(t) = \begin{cases} t & 0 \leq t \leq 1 \\ (2-t) & 1 \leq t \leq 2 \end{cases}$$

$$\therefore F(s) = \frac{1}{1-e^{-2s}} \int_0^2 f(t)e^{-st} dt = \frac{1}{1-e^{-2s}} \left\{ \int_0^1 te^{-st} dt + \int_1^2 (2-t)e^{-st} dt \right\} \equiv \frac{1}{1-e^{-2s}} (I_1 + I_2)$$

where:

$$I_1 = \int_0^1 te^{-st} dt = -\frac{t}{s}e^{-st} \Big|_0^1 + \frac{1}{s} \int_0^1 e^{-st} dt = -\frac{e^{-s}}{s} - \frac{1}{s^2}e^{-st} \Big|_0^1 = -\frac{e^{-s}}{s} - \frac{e^{-s}}{s^2} + \frac{1}{s^2} = \frac{1}{s^2} [1 - e^{-s}(1+s)]$$

(integrating by parts, where: $u = t, dv = e^{-st} dt \Rightarrow du = dt, v = -\frac{1}{s}e^{-st}$)

$$\begin{aligned}
I_2 &= \int_1^2 (2-t)e^{-st} dt = 2 \int_1^2 e^{-st} dt - \int_1^2 te^{-st} dt = -\frac{2}{s}(e^{-2s} - e^{-s}) + \frac{t}{s}e^{-st} \Big|_1^2 - \frac{1}{s} \int_1^2 e^{-st} dt \\
&= \frac{2e^{-s}}{s} [1 - e^{-s}] + \frac{1}{s} (2e^{-2s} - e^{-s}) + \frac{1}{s^2} (e^{-2s} - e^{-s}) \\
&= \frac{2e^{-s}}{s} - \frac{2e^{-2s}}{s} + \frac{2e^{-2s}}{s} - \frac{e^{-s}}{s} + \frac{e^{-2s}}{s^2} - \frac{e^{-s}}{s^2} = \frac{e^{-s}}{s} + \frac{e^{-2s}}{s^2} - \frac{e^{-s}}{s^2}
\end{aligned}$$

Hence:

$$\begin{aligned}
F(s) &= \frac{1}{1-e^{-2s}} \{I_1 + I_2\} = \frac{1}{(1-e^{-s})(1+e^{-s})} \left\{ \left[-\frac{e^{-s}}{s} - \frac{e^{-s}}{s^2} + \frac{1}{s^2} \right] + \left[\frac{e^{-s}}{s} + \frac{e^{-2s}}{s^2} - \frac{e^{-s}}{s^2} \right] \right\} \\
&= \frac{\frac{e^{-2s}}{s^2} - \frac{2e^{-s}}{s^2} + \frac{1}{s^2}}{(1-e^{-s})(1+e^{-s})} = \frac{(e^{-2s} - 2e^{-s} + 1)}{s^2(1-e^{-s})(1+e^{-s})} = \frac{(e^{-s} - 1)^2}{s^2(1-e^{-s})(1+e^{-s})} = \frac{(1-e^{-s})^2}{s^2(1-e^{-s})(1+e^{-s})} \\
&= \frac{(1-e^{-s})}{s^2(1+e^{-s})}
\end{aligned}$$

- **Problem 5a, p 41 (Sheng)**

$$\lim_{t \rightarrow 0^+} f(t) = \lim_{t \rightarrow 0^+} (4 - \cos 3t) = 4 - \lim_{t \rightarrow 0^+} \cos 3t = 4 - 1 = 3$$

$$\lim_{s \rightarrow \infty} sF(s) = \lim_{s \rightarrow \infty} sL\{4 - \cos 3t\} = \lim_{s \rightarrow \infty} s\{4L\{1\} - L\{\cos 3t\}\} = \lim_{s \rightarrow \infty} s\left\{4 \cdot \frac{1}{s} - \frac{s}{s^2+3^2}\right\}$$

$$\lim_{s \rightarrow \infty} \left[4 - \frac{s^2}{s^2+9}\right] = 4 - \lim_{s \rightarrow \infty} \frac{1}{\left(1 + \frac{9}{s^2}\right)} = 4 - 1 = 3$$

Hence the left-hand side & the right-hand side of Thm10 has been verified

- **Problem 6a, p 41 (Sheng)**

$$\lim_{t \rightarrow \infty} [3 - e^{-2t}(\sin 4t - \cos 4t)] = 3 - \lim_{t \rightarrow \infty} (e^{-2t} \sin 4t) + \lim_{t \rightarrow \infty} (e^{-2t} \cos 4t) = 3 - 0 - 0 = 3$$

$$\lim_{s \rightarrow 0} sF(s) = \lim_{s \rightarrow 0} sL\{3 - e^{-2t}(\sin 4t - \cos 4t)\} = \lim_{s \rightarrow 0} s\{3L\{1\} - L\{e^{-2t}(\sin 4t - \cos 4t)\}\} \\ = \lim_{s \rightarrow 0} s\left\{3 \cdot \frac{1}{s} - F(s+2)\right\}$$

(where, according to THM7:

$$F(s) = L\{\sin 4t - \cos 4t\} = \frac{4}{s^2+16} - \frac{s}{s^2+16} = \frac{4-s}{s^2+16} \Rightarrow F(s+2) = \frac{4-(s-2)}{(s+2)^2+16}$$

$$\therefore \lim_{s \rightarrow 0} \left\{3 - \frac{s(6-s)}{(s+2)^2+16}\right\} = 3 - \frac{0}{20} = 3$$

Hence the left-hand side & the right-hand side of Thm11 has been verified

Hints: Assignment I

Problem 1: Keep in mind that the section of $f(t)$ from $\frac{\pi}{2}$ to π is $\sin t$

Problem IIa) First use THM4 to get rid of having to find the LT of that complicated integral. Then use THM5

Problem IIb) Rewrite: First use THM4 as you did in IIa). Then use Euler Thm substitution for $\sin 3t$. Your expression is: $\frac{1}{s} L\left\{t^2 \left(\frac{e^{i3t} - e^{-i3t}}{2i}\right)\right\}$. Use Thm 1 to simplify and then evaluate the integral directly. (See example in pages 5-6 of Handout 3b. Keep in mind, however, in this problem, you'll need to integrate by parts, or you can use Formula 28 Appendix VII (page A7, Sheng)

Problem IIIb) After completing the square in the denominator term, don't forget to also express the numerator term as an expression involving $(s + 4)$: i.e.: $8s - 9 = 8(s + 4) - 41$. Then split up this fraction into two fractions. In the first fraction, the $(s + 4)$ term will of course cancel with the denominator term $(s+4)$. So finding the LT for the first fraction term is straightforward (applying THM7). In the second fraction term:, you'll need to apply THM7 and then THM4.

Problem IVb) To apply THM8, keep in mind that: $e^{2t} = e^6 e^{2(t-3)}$

Problem IVd)
$$F(s) = \frac{1}{(s-4)(s+4)^2} = \frac{1}{(s-4)(s+4)(s+4)} = \frac{1}{(s^2-16)} \cdot \frac{1}{(s+4)}$$

Problem IVe)
$$F(s) = \frac{1}{(s-4)(s+4)^2} = \frac{1}{(s-4)(s-4+8)^2} = \frac{1}{(s-4)[(s-4)+8]^2} = G(s-4)$$

Then :
$$G(s) = \frac{1}{s(s+8)^2} = \frac{1}{s} H(s)$$
 Hence you can apply THM7, THM4.

Problem Vb) (Note typo in Sheng, the quadratic irreducible is raised to the first power, not the second power)

$$\frac{250}{s^3(s^2-4s+5)} = \frac{A_1}{s^3} + \frac{A_2}{s^2} + \frac{A_3}{s} + \frac{B_1s+B_2}{(s^2-4s+5)} \Rightarrow 250 = [A_1 + A_2s + A_3s^2](s^2 - 4s + 5) + (B_1s + B_2)s^3$$

↑(EqnA)

1. Of course, setting $s = 0$ in (Eqn A) $\Rightarrow A_1 = 50$
2. Complete the square on the irreducible in (Eqn A), and then set $s = 2$. You'll end up with a formula: $A_2 = 100 - 2A_3 - 8B_1 - 4B_2$ (Eqn B)
3. The roots of the irreducible are: $s_{1,2} = 2 \pm i$. So setting $s = s_1$ (for instance) in (Eqn A) gives us an equation just in terms of B_1 and B_2 . (I.e., you'll end up with an equation: $250 = -7B_1 + 2B_1 + i(24B_1 + 11B_2)$) Equating real terms on the right hand side with the left hand side, and imaginary terms on the right hand side² with

² On the right hand side, the imaginary terms =0, obviously

those on the left hand side gives two equations and two unknowns. Solving:

$$B_1 = -22, B_2 = 48$$

4. Differentiate both sides of Eqn (A) by s , and then set $s = 0$. You'll get: $A_2 = 40$.
5. Now solving for A_3 in (Eqn B): $A_2 = 22$