

• **Another Example (Fourier Series Methods):**

$$f(x) = \begin{cases} \frac{\pi}{2} + x & \text{for } -\pi \leq x < 0 \\ \frac{\pi}{2} - x & \text{for } 0 \leq x < \pi \end{cases}$$

$$\begin{aligned} a_0 &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) dx = \frac{1}{\pi} \int_{-\pi}^0 \left(\frac{\pi}{2} + x\right) dx + \frac{1}{\pi} \int_0^{\pi} \left(\frac{\pi}{2} - x\right) dx = \frac{1}{2} \int_{-\pi}^{\pi} dx + \frac{1}{\pi} \int_{-\pi}^0 x dx + \frac{1}{\pi} \int_0^{\pi} -x dx = \\ &= \pi + \frac{1}{\pi} \left(-\frac{\pi^2}{2}\right) - \frac{1}{\pi} \left(\frac{\pi^2}{2}\right) = \pi - \pi = 0 \end{aligned}$$

$$\begin{aligned} a_k &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \cos kx dx = \frac{1}{\pi} \int_{-\pi}^0 \left(\frac{\pi}{2} + x\right) \cos kx dx + \frac{1}{\pi} \int_0^{\pi} \left(\frac{\pi}{2} - x\right) \cos kx dx = \\ &\frac{1}{2} \int_{-\pi}^{\pi} \cos kx dx + \frac{1}{\pi} \int_{-\pi}^0 x \cos kx dx + \frac{1}{\pi} \int_0^{\pi} -x \cos kx dx = \\ &\frac{1}{2k} \sin kx \Big|_{-\pi}^{\pi} + \frac{1}{\pi} \left\{ \frac{x}{k} \sin kx \Big|_{-\pi}^0 - \frac{1}{k} \int_{-\pi}^0 \sin kx dx - \frac{x}{k} \sin kx \Big|_0^{\pi} + \frac{1}{k} \int_0^{\pi} \sin kx dx \right\} \\ &= 0 + \frac{1}{\pi} \left\{ 0 + \frac{1}{k^2} \cos kx \Big|_{-\pi}^0 + 0 - \frac{1}{k^2} \cos kx \Big|_0^{\pi} \right\} \end{aligned}$$

The last result warrants careful consideration. The result is *not* always = 0.

$$\begin{aligned} \text{Case 1: } k = 2j &\Rightarrow \cos kx \Big|_{-\pi}^0 = 1 - \cos(2j\pi) = 0 \\ &\cos kx \Big|_0^{\pi} = \cos(2j\pi) - 1 = 0 \end{aligned}$$

$$\begin{aligned} \text{Case 2: } k = 2j + 1 &\Rightarrow \cos kx \Big|_{-\pi}^0 = 1 - \cos((2j + 1)\pi) = 2 \\ &\cos kx \Big|_0^{\pi} = \cos((2j + 1)\pi) - 1 = -2 \end{aligned}$$

$$\text{Hence: } a_k = \frac{1}{\pi} \left\{ \frac{1}{k^2} \cos kx \Big|_{-\pi}^0 - \frac{1}{k^2} \cos kx \Big|_0^{\pi} \right\} = \frac{1}{\pi k^2} \{2 + 2\} = \frac{4}{\pi k^2} \quad (k \text{ odd})$$

$$\text{Or: } a_{2j+1} = \frac{4}{(2j+1)^2 \pi}$$

$$\begin{aligned}
b_k &= \frac{1}{\pi} \int_{-\pi}^{\pi} f(x) \sin kx dx = \frac{1}{\pi} \int_{-\pi}^0 \left(\frac{\pi}{2} + x\right) \sin kx dx + \frac{1}{\pi} \int_0^{\pi} \left(\frac{\pi}{2} - x\right) \sin kx dx = \\
&\frac{1}{2} \int_{-\pi}^{\pi} \sin kx dx + \frac{1}{\pi} \int_{-\pi}^0 x \sin kx dx + \frac{1}{\pi} \int_0^{\pi} -x \sin kx dx = \\
&\frac{1}{2k} \cos kx \Big|_{-\pi}^{\pi} + \frac{1}{\pi} \left\{ \frac{-x}{k} \cos kx \Big|_{-\pi}^0 + \frac{1}{k} \int_{-\pi}^0 \cos kx dx + \frac{x}{k} \cos kx \Big|_0^{\pi} - \frac{1}{k} \int_0^{\pi} \cos kx dx \right\} \\
&= 0 + \frac{1}{\pi} \left\{ \left[\frac{-\pi(2)}{2j+1} \right] - \frac{1}{k^2} \sin kx \Big|_{-\pi}^0 + \left[\frac{\pi(-2)}{2j+1} \right] + \frac{1}{k^2} \sin kx \Big|_0^{\pi} \right\} \\
&= -\frac{4\pi}{(2j+1)}
\end{aligned}$$

$$\begin{aligned}
\text{Hence: } f(x) &= \sum_{j=0}^{\infty} \frac{4}{(2j+1)^2 \pi} \cos[(2j+1)x] - \sum_{j=1}^{\infty} \frac{4\pi}{(2j+1)} \sin[(2j+1)x] \\
&= 4 \sum_{j=1}^{\infty} \left\{ \frac{\cos[(2j+1)x]}{(2j+1)^2 \pi} - \frac{\pi \sin[(2j+1)x]}{(2j+1)} \right\}
\end{aligned}$$

Fourier Series (Complex Form)

$$f(t+p) = f(t) \Rightarrow f(t) = \sum_{k=-\infty}^{\infty} C_k e^{ik\omega t}, (\omega \equiv \frac{2\pi}{p})$$

$$\text{where: } C_k = \frac{1}{p} \int_0^p f(t) e^{-ik\omega t} dt$$

- **Example (Problem 2, p 119):**

$$f(t) = t, 0 < t < 2\pi$$

$$f(t) = \sum_{k=-\infty}^{\infty} C_k e^{ik\omega t}, (\omega \equiv \frac{2\pi}{p} = \frac{2\pi}{2\pi} = 1)$$

$$\begin{aligned}
C_k &= \frac{1}{p} \int_0^p f(t) e^{-ik\omega t} dt = \frac{1}{2\pi} \int_0^{2\pi} t e^{-ikt} dt = \frac{1}{2\pi} \left\{ \frac{i}{k} t e^{-ikt} \Big|_0^{2\pi} - \frac{i}{k} \int_0^{2\pi} e^{-ikt} dt \right\} = \frac{i}{2\pi k} \left\{ 2\pi e^{-2\pi k} - 0 \right\} + \frac{1}{ik} e^{-ikt} \Big|_0^{2\pi} \\
&= \frac{i}{2\pi k} \left\{ 2\pi + \frac{1}{ik} (1-1) \right\} = \frac{i}{k}
\end{aligned}$$

$$\therefore f(t) = \sum_{k=-\infty}^{\infty} \frac{i}{k} e^{ikt} = i \sum_{k=-\infty}^{\infty} \frac{1}{k} e^{ikt} = \sum_{k=-\infty}^{\infty} \frac{1}{k} \left(e^{ik(t+\frac{\pi}{2})} \right)$$

So:

$$a_k = C_k + C_k^* = \frac{i}{k} - \frac{i}{k} = 0$$

$$b_k = i(C_k - C_k^*) = i\left(\frac{i}{k} + \frac{i}{k}\right) = -\frac{2}{k}$$

Proof:

$$a_k = \frac{2}{p} \int_{-\frac{p}{2}}^{\frac{p}{2}} f(t) \cos \omega kt dt = \frac{2}{2\pi} \int_0^{2\pi} t \cos kt dt = \frac{1}{\pi k^2} \int_{u(0)}^{u(2\pi)} u \cos u du, \text{ where } (u = kt)$$

$$= \frac{1}{\pi k^2} \int_0^{2k\pi} u \cos u du = \frac{1}{\pi k^2} \left\{ \cos u \Big|_0^{2k\pi} + u \sin u \Big|_0^{2k\pi} \right\} = \frac{1}{k^2 \pi} [(1-1) + 2k\pi \sin 2k\pi - 0]$$

$$= 0$$

$$b_k = \frac{2}{p} \int_{-\frac{p}{2}}^{\frac{p}{2}} f(t) \sin \omega kt dt = \frac{2}{2\pi} \int_0^{2\pi} t \sin kt dt = \frac{1}{\pi k^2} \int_{u(0)}^{u(2\pi)} u \sin u du, \text{ where } (u = kt)$$

$$= \frac{1}{\pi k^2} \int_0^{2k\pi} u \sin u du = \frac{1}{\pi k^2} \left\{ \sin u \Big|_0^{2k\pi} - u \cos u \Big|_0^{2k\pi} \right\} = -\frac{1}{k^2 \pi} [2k\pi \cos 2k\pi - 0]$$

$$= -\frac{2k\pi}{k^2 \pi} = -\frac{2}{k}$$

• **Example¹ (Problem 4, p 119):**

$$f(t) = t^2, 0 < t < 2\pi, f(t + 2\pi) = f(t)$$

$$f(t) = \sum_{k=-\infty}^{\infty} C_k e^{ik\omega t}, (\omega \equiv \frac{2\pi}{p} = \frac{2\pi}{2\pi} = 1)$$

$$C_k = \frac{1}{p} \int_0^p f(t) e^{-ik\omega t} dt = \frac{1}{2\pi} \int_0^{2\pi} t^2 e^{-ikt} dt = \frac{1}{(2\pi)} \int_0^{2\pi} t^2 e^{-ikt} dt$$

u	dv
t^2 (+)	$e^{-ikt} dt$
$2t$ (-)	$\frac{i}{k} e^{-ikt}$
2 (+)	$-\frac{1}{k^2} e^{-ikt}$
0	$-\frac{i}{k^3} e^{-ikt}$

¹ Modified, to include amplitude and frequency spectra

$$\begin{aligned}
\therefore C_k &= \frac{1}{(2\pi)} \int_0^{2\pi} t^2 e^{-ikt} dt = \frac{1}{2\pi} \left[\frac{i}{k} t^2 e^{-ikt} + \frac{2t}{k^2} e^{-ikt} - \frac{2}{k^3} e^{-ikt} \right]_0^{2\pi} \\
&= \frac{e^{-ikt}}{2\pi k} \left[it^2 + \frac{2}{k} t - \frac{2}{k^2} \right]_0^{2\pi} = \frac{e^{-2ki\pi}}{2\pi k} \left[4i\pi^2 + \frac{4\pi}{k} - \frac{2}{k^2} \right] - \frac{1}{2\pi k^2} \left[-\frac{2}{k^2} \right] \\
&= \frac{4i\pi^2}{2\pi k} + \frac{4\pi}{2\pi k^2} - \frac{2}{2\pi k^3} + \frac{2}{2\pi k^3} = \frac{2i\pi}{k} + \frac{2}{k^2} = \frac{2}{k^2} (1 + ik\pi) = \frac{2+2ik\pi}{k^2} \\
&(k > 0) \\
C_0 &= \frac{p}{p} \int_0^p f(t) dt = \frac{2\pi}{2\pi} \int_0^{2\pi} t^2 dt = \frac{1}{(2\pi)} \left[\frac{1}{3} t^3 \Big|_0^{2\pi} \right] = \frac{4}{3} \pi^2
\end{aligned}$$

Hence: $f(t) = \sum_{k=-\infty}^{\infty} C_k e^{ikt} = \frac{4}{3} \pi^2 + 2 \sum_{k=\pm n, n \neq 0} \left(\frac{1+ik\pi}{k^2} \right) e^{ikt}$

Amplitude Spectrum:

$$|C_k| = \sqrt{CC^*} = \sqrt{2 \left(\frac{1+ik\pi}{k^2} \right) 2 \left(\frac{1-ik\pi}{k^2} \right)} = \frac{2}{k^2} \sqrt{1+k^2} \pi^2$$

$$\therefore |C_0| = \frac{4}{3} \pi^2 \approx 13.159$$

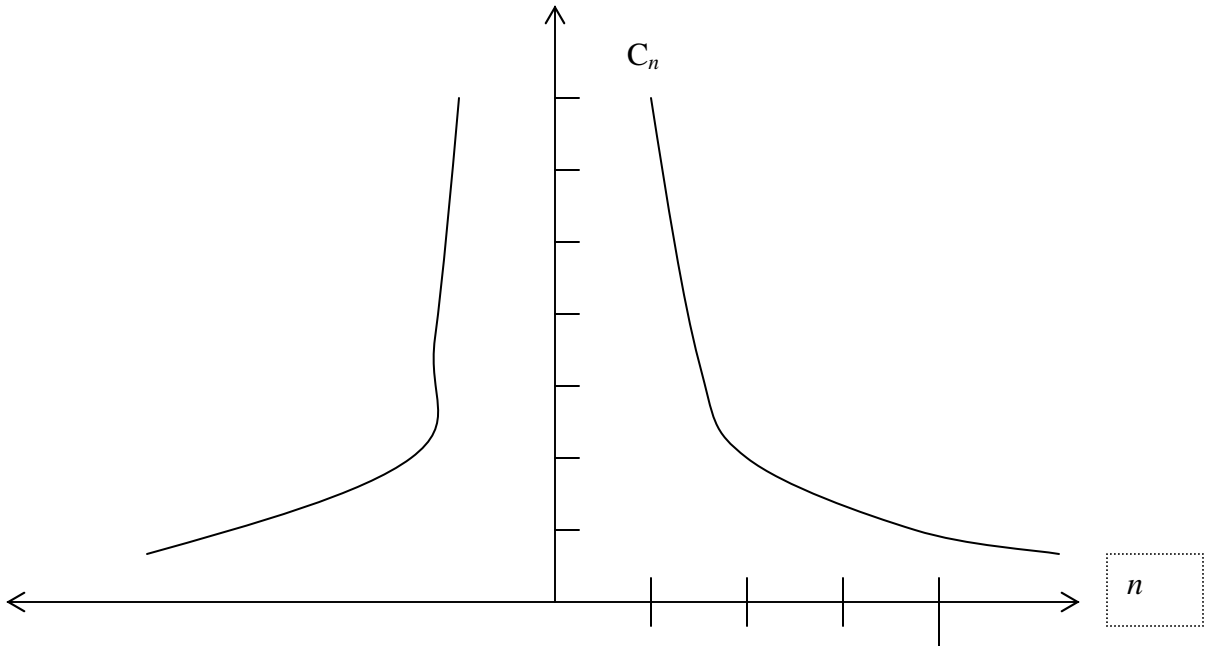
$$|C_{\pm 1}| = 2\sqrt{1+\pi^2} \approx 6.594$$

$$|C_{\pm 2}| = \frac{1}{2} \sqrt{1+4\pi^2} \approx 3.181$$

$$|C_{\pm 3}| = \frac{2}{9} \sqrt{1+9\pi^2} \approx 2.106$$

$$|C_{\pm 4}| = \frac{1}{8} \sqrt{1+16\pi^2} \approx 1.576$$

$$|C_{\pm 5}| = \frac{2}{25} \sqrt{1+25\pi^2} \approx 1.259$$



Frequency Spectrum:

$$\theta_n = \tan^{-1}\left(\frac{B_n}{A_n}\right) = \tan^{-1}\left(\frac{2k\pi/k^2}{2/k^2}\right) = \tan^{-1}(k\pi)$$

$$\theta_0 = \tan^{-1}(0) = 0$$

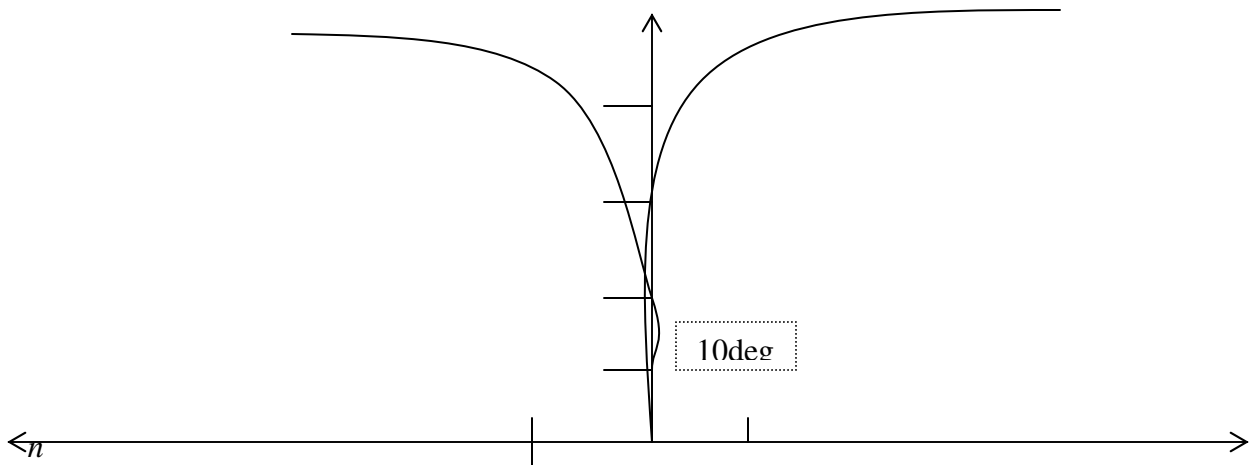
$$\theta_{\pm 1} = \tan^{-1}(\pi) \approx 72.3^\circ$$

$$\theta_{\pm 2} = \tan^{-1}(2\pi) \approx 81^\circ$$

$$\theta_{\pm 3} = \tan^{-1}(3\pi) \approx 84^\circ$$

$$\theta_{\pm 4} = \tan^{-1}(4\pi) \approx 85^\circ$$

$$\theta_{\pm 5} = \tan^{-1}(5\pi) \approx 86^\circ$$



- **Fast Fourier Transform (FFT)**

As evidenced in the above calculations, it's an inefficient procedure to find the Fourier coefficients a_k, b_k from first principles.

Consider instead the following scheme: Suppose we wish to find the Fourier coefficients a_k, b_k for $0 \leq k \leq N$

Define (complex) numbers:
$$c_k = a_k + ib_k = \frac{1}{N} \sum_{r=0}^{2N-1} f(\theta_r) \exp(ik\theta_r) \quad (\text{IX.2})$$

Define integers M, K such that: $2N = MK$. Then, for $0 \leq j_0 < M, 0 \leq r_0 < K$:

$$j = j_1 M + j_0, r = r_1 K + r_0$$

- **Exercise:**

Using the fact that any sum of the form: $\sum_{r=0}^{2m-1} y_r$ can be re-written in the form:

$\sum_{r_0=0}^{K-1} \sum_{r_1=0}^{M-1} y_{rK+r_0}$ show that (IX.2) can be expressed as:

$$c_k = a_k + ib_k = \frac{1}{N} \sum_{r=0}^{K-1} C(j_0, r_0) \exp\left(2\pi i \left(j_1 + \frac{j_0}{M}\right) \frac{r_0}{K}\right)$$

$$\text{where: } C(j_0, r_0) = \sum_{r=0}^{M-1} f(\theta_{r_1K+r_0}) \exp\left(\frac{2\pi i j_0 r_1}{M}\right)$$

which is the FFT algorithm