MATH 360 **Name: KEY**
EXAM I
February 15, 2006

Directions: For maximum credit, please show all work in most reasonable detail. If you run out of room, you may pick up extra paper (supplied by instructor/proctor) and attach to this exam. No books or notes. Formula sheet provided. Calculator permitted, with programming/memory mode shut off. Please choose FOUR from the following SIX (worth 25 pts each.) If you do more, I will grade the best <u>four</u>. Bonus problem included. Good luck!

I. (25 total)

a.) (13) Find $L\{f(t)\}$ **for** $f(t)=\begin{Bmatrix} 0 & 0\le t<1 \\ 3-3t & 1\le t<2 \\ 1 & t\ge 2 \end{Bmatrix}$ **from first principles (i.e., using definition of Laplace Transform)**

$$L\{f(t)\}=\int_0^\infty f(t)e^{-st}dt=\int_0^1 0\cdot e^{-st}dt+\int_1^2 3(1-t)e^{-st}dt+\int_2^\infty 1\cdot e^{-st}dt=3\int_1^2(1-t)e^{-st}dt+\int_2^\infty e^{-st}dt$$

$$=3\int_1^2 e^{-st}dt-3\int_1^2 te^{-st}dt+\int_2^\infty e^{-st}dt=3\cdot\left(-\tfrac{1}{s}e^{-st}\right)\Big|_1^2-3\left\{-\tfrac{t}{s}e^{-st}\Big|_1^2+\tfrac{1}{s}\int_1^2 e^{-st}dt\right\}-\tfrac{1}{s}e^{-st}\Big|_2^\infty$$

(Note: The middle integral is evaluated using integration by parts, where: $U=t,\ dV=e^{-st}dt$)

$$\therefore L\{f(t)\}=-3\tfrac{e^{-st}}{s}\Big|_1^2-3\left\{-\tfrac{t}{s}e^{-st}\Big|_1^2-\tfrac{1}{s^2}e^{-st}\Big|_1^2\right\}-\lim_{d\to\infty}\tfrac{1}{s}e^{-st}\Big|_2^d$$

$$=-3\left(\tfrac{e^{-2s}}{s}-\tfrac{e^{-s}}{s}\right)-3\left\{-\tfrac{2}{s}e^{-2s}+\tfrac{1}{s}e^{-s}-\tfrac{1}{s^2}\left(e^{-2s}-e^{-s}\right)\right\}-\tfrac{1}{s}\lim_{d\to\infty}e^{-sd}+\tfrac{1}{s}e^{-2s}$$

$$=-3\tfrac{e^{-2s}}{s}+3\tfrac{e^{-s}}{s}+6\tfrac{e^{-2s}}{s}-3\tfrac{e^{-s}}{s}+3\tfrac{e^{-2s}}{s^2}-3\tfrac{e^{-s}}{s^2}-0+\tfrac{1}{s}e^{-2s}$$

$$=e^{-2s}\left[-\tfrac{3}{s}+\tfrac{6}{s}+\tfrac{3}{s^2}+\tfrac{1}{s}\right]+e^{-s}\left[\tfrac{3}{s}-\tfrac{3}{s}-\tfrac{3}{s^2}\right]=\tfrac{e^{-s}}{s^2}\left[-3+e^{-s}(4s+3)\right]$$

b.) (12) Verify your answer in a.) using step function method

$$f(t)=\begin{Bmatrix} 0 & 0\le t<1 \\ 3-3t & 1\le t<2 \\ 1 & 2\le t \end{Bmatrix}=0\cdot[u(t)-u(t-1)]+3(1-t)[u(t-1)-u(t-2)]+1\cdot u(t-2)$$

$$=-3(t-1)u(t-1)-3u(t-2)+3tu(t-2)+u(t-2)=-3(t-1)u(t-1)-2u(t-2)+3tu(t-2)$$

$$\therefore L\{f(t)\} = -3L\{(t-1)u(t-1)\} - 2L\{u(t-2)\} + 3L\{tu(t-2)\} = -3L\{u(t-1)(t-1)\} - 2\tfrac{e^{-2s}}{s} + 3(-1)\tfrac{d}{ds}L\{u(t-2)\}$$
$$= -3e^{-s}L\{t\} - 2\tfrac{e^{-2s}}{s} - 3\tfrac{d}{ds}\left(\tfrac{e^{-2s}}{s}\right) = -3e^{-s}\left(\tfrac{1}{s^2}\right) - 2\tfrac{e^{-2s}}{s} - 3\left(\tfrac{-s2e^{-2s} - e^{-2s}}{s^2}\right) = -3\tfrac{e^{-s}}{s^2} - 2\tfrac{e^{-2s}}{s} + 6\tfrac{e^{-2s}}{s} + 3\tfrac{e^{-2s}}{s^2}$$
$$= \tfrac{e^{-s}}{s^2}\left[-3 + e^{-s}(4s+3)\right]$$

II (25 total) Given $f(t) = te^{-t}\sinh t$

a.) (15) Find $L\{f(t)\}$ by inserting definition of hyperbolic sine:
$\sinh at = \frac{1}{2}\left(e^{at} - e^{-at}\right)$ **and using THM5**

$$f(t) = te^{-t}\tfrac{1}{2}\left(e^{t} - e^{-t}\right) = \tfrac{t}{2}\left(1 - e^{-2t}\right)$$
$$\therefore L\{f(t)\} = L\left\{t \cdot \tfrac{1}{2}\left(1 - e^{-2t}\right)\right\} = \tfrac{1}{2}L\left\{t \cdot \left(1 - e^{-2t}\right)\right\} = \tfrac{1}{2}(-1)^1\tfrac{d}{ds}L\left\{\left(1 - e^{-2t}\right)\right\}$$
$$= -\tfrac{1}{2}\tfrac{d}{ds}\left(\tfrac{1}{s} - \tfrac{1}{s+2}\right) = -\tfrac{1}{2}\tfrac{d}{ds}\left(\tfrac{s+2-s}{s(s+2)}\right) = -\tfrac{1}{2}\tfrac{d}{ds}\left(\tfrac{2}{(s^2+2s)}\right) = -\tfrac{d}{ds}\left(\tfrac{1}{(s^2+2s)}\right) = -\left(\tfrac{-(2s+2)}{(s^2+2s)^2}\right)$$
$$= \tfrac{2(s+1)}{s^2(s+2)^2}$$

b.) (10) Find $L\{f(t)\}$ by *not* inserting definition of hyperbolic sine, but by using THM7 and THM5. Show that you get the same answer

- According to THM7: $L\{f(t)\} = L\left\{e^{-t}t\sinh t\right\} = F(s+1)$, where:
  $F(s) = L\{t\sinh t\}$
- According to THM5:
  $L\{t\sinh t\} = (-1)\frac{d}{ds}L\{\sinh t\} = (-1)\frac{d}{ds}\left(\frac{1}{s^2-1}\right) = -\left(\frac{-2s}{(s^2-1)^2}\right) = \frac{2s}{(s^2-1)^2}$
- So: $F(s+1) = \frac{2s+2}{((s+1)^2-1)^2} = \frac{2(s+1)}{(s^2+2s+1-1)^2} = \frac{2(s+1)}{(s^2+2s)^2} = \frac{2(s+1)}{s^2(s+2)^2}$

III. a.) (10) Find $L^{-1}\left[\frac{e^{-3s}}{(s+3)^4}\right]$

- According to THM8: $F(s) = e^{-3s}\left(\frac{1}{(s+3)^4}\right) = e^{-3s}G(s) = L\{u(t-3)f(t-3)\}$, where:
  $G(s) = \frac{1}{(s+3)^4}$
- According to THM7: $G(s) = \frac{1}{(s+3)^4} = H(s+3) = L\left\{e^{-3t}h(t)\right\}$, where:
  $H(s) = L\{h(t)\} = \frac{1}{s^4}$
- So:

$$H(s) = L\{h(t)\} = \tfrac{1}{s^4} = \tfrac{1}{3!}\cdot\tfrac{3!}{s^4} \Rightarrow h(t) = \tfrac{1}{6}t^3 \Rightarrow G(s) = L\left\{\tfrac{1}{6}e^{-3t}t^3\right\} \Rightarrow$$
$$F(s) = L\left\{\tfrac{1}{6}u(t-3)e^{-3(t-3)}(t-3)^3\right\} \Rightarrow f(t) = \tfrac{1}{6}u(t-3)e^9e^{-3t}(t-3)^3$$
$$\therefore f(t) = \tfrac{1}{6}e^9e^{-3t}(t-3)^3, t > 3$$

c.) **(15)** $L^{-1}\left[\frac{1}{s}\frac{s+3}{(s+3)^2+16}\right]$

- According to THM4: $\frac{1}{s}\frac{(s+3)}{(s+3)^2+4^2} = L\left\{\int_0^t f(\omega)d\omega\right\}$, where: $L\{f(t)\} = \frac{(s+3)}{(s+3)^2+4^2}$
- According to THM7: $L\{f(t)\} = \frac{(s+3)}{(s+3)^2+4^2} = F(s+3) = L\{e^{-3t}g(t)\}$, where: $L\{g(t)\} = \frac{s}{s^2+4^2} = \cos 4t$
- So:
$$f(t) = L^{-1}\left(\frac{1}{s}\cdot\frac{s+3}{(s+3)^2+4}\right) = \int_0^t e^{-3\omega}\cos 4\omega d\omega = \frac{e^{-3\omega}}{3^2+4^2}\left(-3\cos 4\omega + 4\sin 4\omega\right)\Big|_0^t$$
$$= \frac{e^{-3t}}{25}(-3\cos 4t + 4\sin 4t) - \frac{1}{25}(-3) = \frac{1}{25}\left[e^{-3t}(4\sin 4t - 3\cos 4t) + 3\right]$$

IV. Given your answer for *f*(*t*) in problem Ib.)

a.) (10) Find $f'(t)$

From Ib**:** $f(t) = \begin{Bmatrix} 0 & 0 \le t < 1 \\ 3-3t & 1 \le t < 2 \\ 1 & 2 \le t \end{Bmatrix} = 3(1-t)u(t-1) - 2u(t-2) + 3tu(t-2)$

So:
$$f'(t) = [-3u(t-1) + 3(1-t)\delta(t-1)] - 2\delta(t-2) + 3[u(t-2) + t\delta(t-2)]$$
$$= 3(u(t-2) - u(t-1)) + 3(1-t)\delta(t-1) + \delta(t-2)(3t-2)$$

b.) (15) Find $L\{f'(t)\}$

$$L\{f'(t)\} = 3L\{u(t-2)\} - 3L\{u(t-1)\} + 3L\{\delta(t-1)\} - 3L\{t\delta(t-1)\} + 3L\{t\delta(t-2)\} - 2L\{\delta(t-2)\}$$

$$= 3\frac{e^{-2s}}{s} - 3\frac{e^{-s}}{s} + 3e^{-s} - 3(-1)\frac{d}{ds}\left(e^{-s}\right) + 3(-1)\frac{d}{ds}\left(e^{-2s}\right) - 2e^{-2s}$$

$= 3\frac{e^{-2s}}{s} - 3\frac{e^{-s}}{s} + 3e^{-s} + 3\left(e^{-s}\right) + 3\left(2e^{-2s}\right) - 2e^{-2s}$

$= \frac{3e^{-2s}}{s} - \frac{3e^{-s}}{s} + 3e^{-s} + 3e^{-s} + 6e^{-2s} - 2e^{-2s}$

$= e^{-s}\left[6 - \frac{3}{s} + e^{-s}\left(4 - \frac{3}{s}\right)\right]$

V. (25) Given *f*(*t*) *f*(*t*) = (*t* + 4)^{2}cosh4*t*

a.) (5) Find $\lim_{t\to 0^+} f(t)$

$\lim_{t\to 0}(t+4)^2 \cosh 4t = \left[\lim_{t\to o}(t+4)^2\right]\left[\lim_{t\to 0} \cosh 4t\right]$

$= 16 \cdot \frac{1}{2}\left(e^0 + e^0\right) = 16$

b.) (15) **Find** $F(s) = L\{f(t)\}$

$L\{(t+4)^2 \cosh 4t\} = \frac{1}{2} L\{e^{4t}(t+4)^2\} + \frac{1}{2} L\{e^{-4t}(t+4)^2\}$

- According to THM7: $L\{e^{\pm 4t}(t+4)^2\} = F(s \mp 4)$, where: $F(s) = L\{(t+4)^2\}$
  $= L\{t^2 + 2t + 16\} = \frac{2}{s^3} + \frac{2}{s^2} + \frac{16}{s}$

  Hence:
  $\frac{1}{2} L\{e^{4t}(t+4)^2\} + \frac{1}{2} L\{e^{-4t}(t+4)^2\} = \frac{1}{2}\left\{\frac{2}{(s-4)^3} + \frac{2}{(s-4)^2} + \frac{16}{(s-4)} + \frac{2}{(s+4)^3} + \frac{2}{(s+4)^2} + \frac{16}{(s+4)}\right\}$
  $= \frac{1}{(s-4)}\left\{\frac{1}{(s-4)^2} + \frac{1}{(s-4)} + 8\right\} + \frac{1}{(s+4)}\left\{\frac{1}{(s+4)^2} + \frac{1}{(s+4)} + 8\right\}$

(5) Find $\lim_{s\to\infty} sF(s)$**, thereby verifying THM10**

$\lim_{s\to\infty} sF(s) = \lim_{s\to\infty}\left\{\frac{s}{(s-4)^3} + \frac{s}{(s-4)^2} + \frac{8s}{(s-4)} + \frac{s}{(s+4)^3} + \frac{s}{(s+4)^2} + \frac{8s}{(s+4)}\right\}$

$= 0 + 0 + \lim_{s\to\infty}\left\{\frac{8s}{s-4} + \frac{8s}{s+4}\right\} + 0 + 0 = 8\lim_{s\to\infty}\left(\frac{1}{1-\frac{4}{s}} + \frac{1}{1+\frac{4}{s}}\right) = 8(1+1) = 16$

VI(25pts) Prove

a) (10) $\frac{d}{dt}u(t-t_0) = \delta(t-t_0)$

$\frac{d}{dt}u(t-t_0) = \delta(t-t_0)$

Proof: $\frac{d}{dt}u(t-t_0) = \lim_{h\to 0}\frac{u(t-t_0+h) - u(t-t_0)}{h}$

Define: *h* = -*a*. Then:

$$\tfrac{d}{dt}u(t-t_0)=\lim_{a\to 0}\frac{u(t-t_0-a)-u(t-t_0)}{-a}=\lim_{a\to 0}\frac{u(t-t_0)-u(t-t_0-a)}{a}=\delta(t-t_0)$$

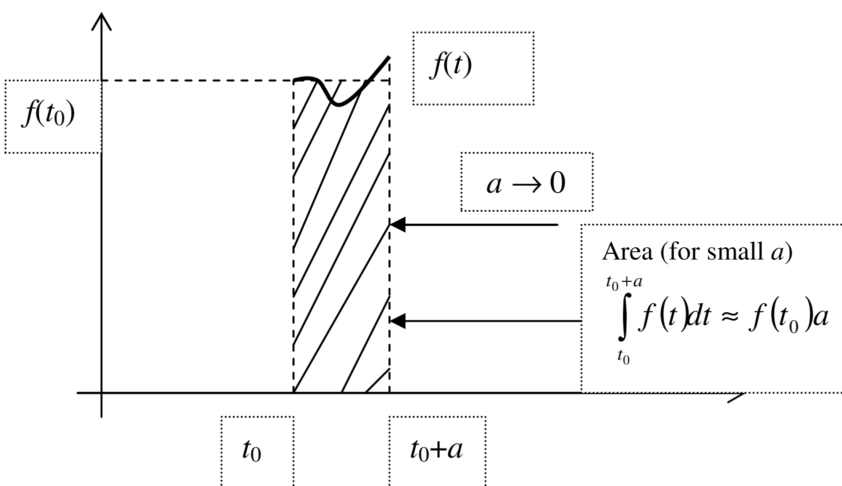
b) (15) $\int_{-\infty}^{\infty} f(t)\delta(t-t_0)=f(t_0)$

$$\int_{-\infty}^{\infty} f(t)\delta(t-t_0)=\int_{-\infty}^{\infty} f(t)\lim_{a\to 0}\left[\frac{u(t-t_0)-u(t-t_0-a)}{a}\right]$$

$$=\lim_{a\to 0}\tfrac{1}{a}\left\{\int_{-\infty}^{\infty} f(t)u(t-t_0)dt-\int_{-\infty}^{\infty} f(t)u(t-t_0-a)dt\right\}=\lim_{a\to 0}\tfrac{1}{a}\left\{\int_{t_0}^{\infty} f(t)dt-\int_{t_0+a}^{\infty} f(t)dt\right\}$$

$$=\lim_{a\to 0}\tfrac{1}{a}\int_{t_0}^{t_0+a} f(t)dt=\lim_{a\to 0}\tfrac{1}{a}\left[f(t_0)a\right]=f(t_0)$$

The last step in the proof involved approximating the integral $\int_{t_0}^{t_0+a} f(t)dt$ with the area of a rectangle of height $f(t_0)$ and width a. Note that in the limit $a\to 0$, the approximation becomes exact. (See figure next page) :



BONUS (20) PTS The Gamma Function

a) (15)Prove: (Property 2): $\Gamma(1/2) = \sqrt{\pi}$

For $\Gamma(1/2)$ =1, then $p = -{}^{1}/_{2}$

$$\Gamma\left(\tfrac{1}{2}\right) = \int_0^\infty x^{-\frac{1}{2}} e^{-x} dx \Rightarrow u = x^{\frac{1}{2}}, du = \tfrac{1}{2} x^{-\frac{1}{2}} dx, x = u^2 \Rightarrow dx = 2x^{\frac{1}{2}} du = 2udu$$

$$\therefore \int_0^\infty x^{-\frac{1}{2}} e^{-x} dx = \int_{u(0)}^{u(\infty)} u^{-1} e^{-u^2} (2udu) = 2\int_0^\infty e^{-u^2} du$$

This last integral is evaluated by performing the following trick:

$$\Gamma\left(\tfrac{1}{2}\right) = 2\int_0^\infty e^{-u^2} du \Rightarrow \left[\Gamma\left(\tfrac{1}{2}\right)\right]^2 = 4\int_0^\infty\int_0^\infty e^{-u^2} e^{-w^2} dudw = \int_{-\infty}^\infty\int_{-\infty}^\infty e^{-\left(u^2+w^2\right)} dudw$$

(I.e. by squaring the integral we convert it into a double integral over the first quadrant of the Cartesian plane, which is one-fourth the (infinite) region of the entire Cartesian Plane.) The integral can be evaluated using a change of variables from Cartesian (u,w) to Polar Coordinates (r,θ):

$$\int_{-\infty}^\infty\int_{-\infty}^\infty e^{-\left(u^2+w^2\right)} dudw = \int_0^{2\pi}\int^\infty e^{-r^2} rdrd\theta = \int_0^{2\pi} d\theta \int_0^\infty e^{-r^2} rdr = 2\pi\left\{-\tfrac{1}{2} e^{-r^2}\Big|_0^\infty\right\}$$

(The integral with the exponential term is evaluated using the u – subsitution: $u = r^2$)

Hence:

$$\left[\Gamma\left(\tfrac{1}{2}\right)\right]^2 = 2\pi\left\{-\tfrac{1}{2}\left[\lim_{d\to\infty} e^{-d^2} - e^0\right]\right\} = -\pi[0-1] = \pi$$
$$\therefore \Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}$$

b) (5) Find $L\{e^{-5t}(t-5)^{5/2}\}$

Let $u = t - 5$, then: $L\{e^{-5t}(t-5)^{5/2}\} = L\{e^{25}e^{-5u}\, u^{5/2}\} = e^{25}L\{e^{-5u}\, u^{5/2}\}$

$= e^{25}L\left\{e^{-5u}u^{5/2}\right\} = e^{25}F(s+5)$, where: $F(s) = L\left\{u^{5/2}\right\} = \frac{\Gamma\left(\frac{5}{2}\right)}{s^{\frac{7}{2}}}$

Where: $\Gamma\left(\tfrac{5}{2}\right) = \tfrac{5}{2}\Gamma\left(\tfrac{3}{2}\right) = \tfrac{5}{2}\cdot\tfrac{3}{2}\Gamma\left(\tfrac{1}{2}\right) = \frac{15\sqrt{\pi}}{4}$ (According to Property 2, 3)

So: $F(s) = L\{u^{5/2}\} = \frac{\Gamma\left(\frac{5}{2}\right)}{s^{\frac{7}{2}}} = \frac{15\sqrt{\pi}}{4} s^{-\frac{7}{2}} \Rightarrow L\{f(t)\} = e^{25} \frac{15\sqrt{\pi}}{4} (s+5)^{-\frac{7}{2}}$