MA360 **Handout 9b (week of March 20)**
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Assignment: Problems 1,3,5,7,9 p108 Sheng

Fourier Analysis & Fourier Synthesis

Consider any periodic function, i.e., a function f with property: $f(t + p) = f(p)$ (for some finite, nonzero p). Then f can be represented in terms of the following series:

$$f(t) = \sum_{n=0}^{\infty} \left\{ a_n \cos\left(\tfrac{2n\pi}{p}\right) + b_n \sin\left(\tfrac{2n\pi}{p}\right) \right\} \qquad \text{Eqn (9b.1)}$$

where: $a_n = \frac{2}{p}\int_{-\frac{p}{2}}^{\frac{p}{2}} f(t)\cos\left(\frac{2n\pi}{p}\right)dt$, $b_n = \frac{2}{p}\int_{-\frac{p}{2}}^{\frac{p}{2}} f(t)\sin\left(\frac{2n\pi}{p}\right)dt$ (for $n = 0,1,2,\ldots$)

- **Remark1** Eqn (9b.1) can be understood as a representation of any periodic function in terms of a series of elementary periodic functions, much like a Taylor series can represent any differentiable function in terms of a series of elementary powers. But there's something else, however, concerning the Fourier series: If one considers the (infinite) set: $\left\{\cos\left(\frac{2n\pi}{p}\right), \sin\left(\frac{2n\pi}{p}\right) \mid n = 0,1,2,\ldots\right\}$ of these elementary functions (indexed according to nonnegative n) then these elements are "orthogonal" in the sense that their "dot product[1]" is always zero, except in the case when $n = m$. In fact:

$$\cos\left(\tfrac{2n\pi}{p}\right)\bullet\cos\left(\tfrac{2m\pi}{p}\right) \equiv \int_{-\frac{p}{2}}^{\frac{p}{2}} \cos\left(\tfrac{2n\pi}{p}\right)\cos\left(\tfrac{2m\pi}{p}\right)dt = \tfrac{p}{2}\delta_{nm}, \text{ where: } \delta_{nm} = \begin{cases} 0 & n \neq m \\ 1 & n = m \end{cases}$$

$$\sin\left(\tfrac{2n\pi}{p}\right)\bullet\sin\left(\tfrac{2m\pi}{p}\right) \equiv \int_{-\frac{p}{2}}^{\frac{p}{2}} \sin\left(\tfrac{2n\pi}{p}\right)\sin\left(\tfrac{2m\pi}{p}\right)dt = \tfrac{p}{2}\delta_{nm}$$

$$\sin\left(\tfrac{2n\pi}{p}\right)\bullet\cos\left(\tfrac{2m\pi}{p}\right) \equiv \int_{-\frac{p}{2}}^{\frac{p}{2}} \sin\left(\tfrac{2n\pi}{p}\right)\cos\left(\tfrac{2m\pi}{p}\right)dt = 0 \text{ (for all nonnegative } n\text{ , } m\text{)}$$

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[1] The "dot product" for two elements in this infinite dimensional vector space, spanned by such elementary functions, is defined by an *integral*: $f \bullet g = \int_{-p/2}^{p/2} f(t)g(t)dt$, for any f, g such that: $f(t + p) = f(t)$, $g(t + p) = g(t)$. The type of vector space, defined in this particular example with unit spanning vectors: $\left\{\sin\left(\frac{2n\pi}{p}\right), \cos\left(\frac{2n\pi}{p}\right)\right\}$, is an example of a *Hilbert Space.*

To verify the first two results, consider the case when $n = m$:

$$\int_{-\frac{p}{2}}^{\frac{p}{2}} \cos\left(\tfrac{2n\pi t}{p}\right)\cos\left(\tfrac{2n\pi t}{p}\right)dt = \int_{-\frac{p}{2}}^{\frac{p}{2}} \cos^2\left(\tfrac{2n\pi t}{p}\right)dt = \tfrac{1}{2}\int_{-\frac{p}{2}}^{\frac{p}{2}} \left(1+\cos\left(\tfrac{4n\pi t}{p}\right)\right)dt$$

$$= \tfrac{1}{2}\int_{-\frac{p}{2}}^{\frac{p}{2}} dt + -\tfrac{1}{2}\int_{-p/2}^{p/2} \cos\left(\tfrac{4n\pi t}{p}\right) = \tfrac{1}{2}\left\{t\Big|_{-p/2}^{p/2} - \tfrac{p}{4n\pi}\sin\left(\tfrac{4n\pi t}{p}\right)\Big|_{-p/2}^{p/2}\right\}$$

$$= \tfrac{1}{2}\left\{p - \tfrac{p}{4n\pi}\left(\sin 2n\pi + \sin 2n\pi\right)\right\} = \tfrac{p}{2}$$

…and the case when $n \neq m$:

$$\int_{-\frac{p}{2}}^{\frac{p}{2}} \cos\left(\tfrac{2n\pi t}{p}\right)\cos\left(\tfrac{2m\pi t}{p}\right)dt = \tfrac{1}{2}\int_{-p/2}^{p/2}\left[\cos(n-m)\tfrac{2\pi t}{p} + \cos(n+m)\tfrac{2\pi t}{p}\right]dt$$

$$= \tfrac{1}{2}\left\{\tfrac{p}{2\pi(n-m)}\sin((n-m)\tfrac{2\pi t}{p})\Big|_{-p/2}^{p/2} + \tfrac{p}{2\pi(n+m)}\sin(n+m)\tfrac{2\pi t}{p}\Big|_{-p/2}^{p/2}\right\}$$

$$= \tfrac{1}{2}\left\{\tfrac{p}{2\pi(n-m)}\left[\sin(n-m)\pi + \sin(n-m)\pi\right] + \tfrac{p}{2\pi(n+m)}\left[\sin(n+m)\pi - \sin(n+m)\pi\right]\right\}$$

$$= 0$$

By the same methods, it can also be established that:

$$\sin\left(\tfrac{2n\pi t}{p}\right)\bullet\sin\left(\tfrac{2m\pi t}{p}\right) \equiv \int_{-\frac{p}{2}}^{\frac{p}{2}} \sin\left(\tfrac{2n\pi t}{p}\right)\sin\left(\tfrac{2m\pi t}{p}\right)dt = \tfrac{p}{2}\delta_{nm}$$

Moreover, for *any* nonnegative n , m:

$\sin\left(\tfrac{2n\pi t}{p}\right)\bullet\cos\left(\tfrac{2m\pi t}{p}\right) \equiv \int_{-\frac{p}{2}}^{\frac{p}{2}} \sin\left(\tfrac{2n\pi t}{p}\right)\cos\left(\tfrac{2m\pi t}{p}\right)dt = 0$, since (in the case $n \neq m$) :

$$\int_{-\frac{p}{2}}^{\frac{p}{2}} \sin\left(\tfrac{2n\pi t}{p}\right)\cos\left(\tfrac{2m\pi t}{p}\right)dt = \tfrac{1}{2}\int_{-p/2}^{p/2}[\sin\left((n+m)\tfrac{2\pi t}{p}\right) + \sin\left((n-m)\tfrac{2\pi t}{p}\right)]dt$$

$$= \tfrac{1}{2}\left\{-\tfrac{p}{2\pi(n+m)}\cos\left((n+m)\tfrac{2\pi t}{p}\right)\Big|_{-p/2}^{p/2} - \tfrac{p}{2\pi(n-m)}\cos\left((n-m)\tfrac{2\pi t}{p}\right)\Big|_{-p/2}^{p/2}\right\}$$

$$= -\tfrac{p}{4\pi(n+m)}\left[\cos(n+m)\pi - \cos(-(n+m)\pi)\right] - \tfrac{p}{4\pi(n-m)}\left[\cos((n-m)\pi) - \cos(-(n-m)\pi)\right]$$

$$= 0$$

…since cos(-A) = cos(A) for any A, i.e. cosine is an even function.

In the case when $n = m$:

$$\sin\left(\tfrac{2n\pi t}{p}\right)\bullet\cos\left(\tfrac{2n\pi t}{p}\right)\equiv\int_{-\frac{p}{2}}^{\frac{p}{2}}\sin\left(\tfrac{2n\pi t}{p}\right)\cos\left(\tfrac{2n\pi t}{p}\right)dt=\tfrac{p}{4\pi n}\sin^2\left(\tfrac{2\pi nt}{p}\right)\Big|_{-p/2}^{p/2}$$

$$=\tfrac{p}{4\pi n}\left\{\sin^2(n\pi)-\sin^2(n\pi)\right\}=0$$

- **Remark 2**: Using the above orthogonality properties of the set of elementary periodic functions $\left\{\cos\left(\tfrac{2n\pi}{p}\right),\sin\left(\tfrac{2n\pi}{p}\right)\mid n=0,1,2,\ldots\right\}$, the expansion coefficients in (Eqn (9b.1)) can be derived:

$$\int_{-\frac{p}{2}}^{\frac{p}{2}}f(t)\cos\left(\tfrac{2n\pi t}{p}\right)dt=\int_{-p/2}^{p/2}\sum_{m=0}^{\infty}\left(a_m\cos\left(\tfrac{2\pi mt}{p}\right)+b_m\sin\left(\tfrac{2\pi mt}{p}\right)\right)\cos\left(\tfrac{2\pi nt}{p}\right)dt$$

$$=\sum_{m=0}^{\infty}a_m\int_{-p/2}^{p/2}\cos\left(\tfrac{2m\pi t}{p}\right)\cos\left(\tfrac{2n\pi t}{p}\right)dt+\sum_{m=0}^{\infty}b_m\int_{-p/2}^{p/2}\sin\left(\tfrac{2\pi mt}{p}\right)\cos\left(\tfrac{2\pi nt}{p}\right)dt$$

$$=\sum_{m=0}^{\infty}a_m\tfrac{p}{2}\delta_{mn}+0=\tfrac{p}{2}a_n\Rightarrow a_n=\tfrac{2}{p}\int_{-p/2}^{p/2}f(t)\cos\left(\tfrac{2n\pi t}{p}\right)dt$$

$$\int_{-\frac{p}{2}}^{\frac{p}{2}}f(t)\sin\left(\tfrac{2n\pi t}{p}\right)dt=\int_{-p/2}^{p/2}\sum_{m=0}^{\infty}\left(a_m\cos\left(\tfrac{2\pi mt}{p}\right)+b_m\sin\left(\tfrac{2\pi mt}{p}\right)\right)\sin\left(\tfrac{2\pi nt}{p}\right)dt$$

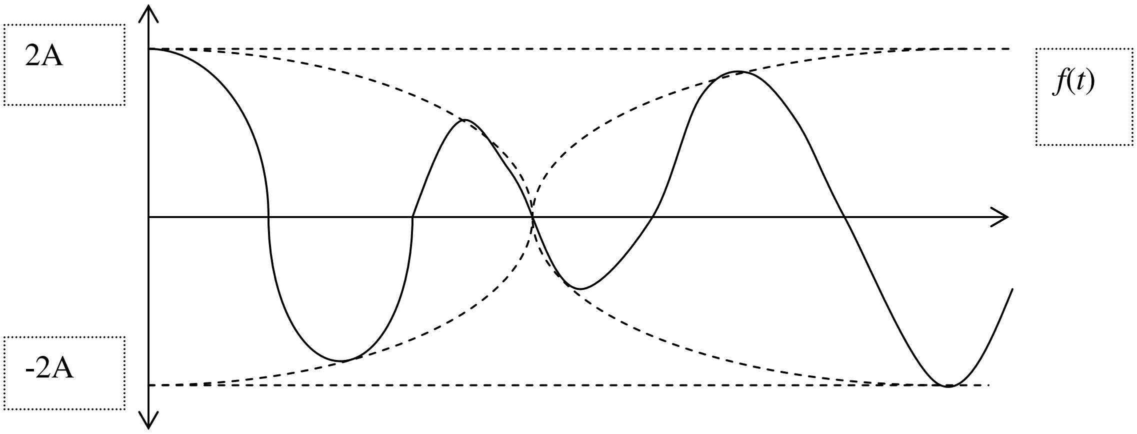
$$=\sum_{m=0}^{\infty}a_m\int_{-p/2}^{p/2}\cos\left(\tfrac{2m\pi t}{p}\right)\sin\left(\tfrac{2n\pi t}{p}\right)dt+\sum_{m=0}^{\infty}b_m\int_{-p/2}^{p/2}\sin\left(\tfrac{2\pi mt}{p}\right)\sin\left(\tfrac{2\pi nt}{p}\right)dt$$

$$=0+\sum_{m=0}^{\infty}b_m\tfrac{p}{2}\delta_{mn}=\tfrac{p}{2}b_n\Rightarrow b_n=\tfrac{2}{p}\int_{-p/2}^{p/2}f(t)\cos\left(\tfrac{2n\pi t}{p}\right)dt$$

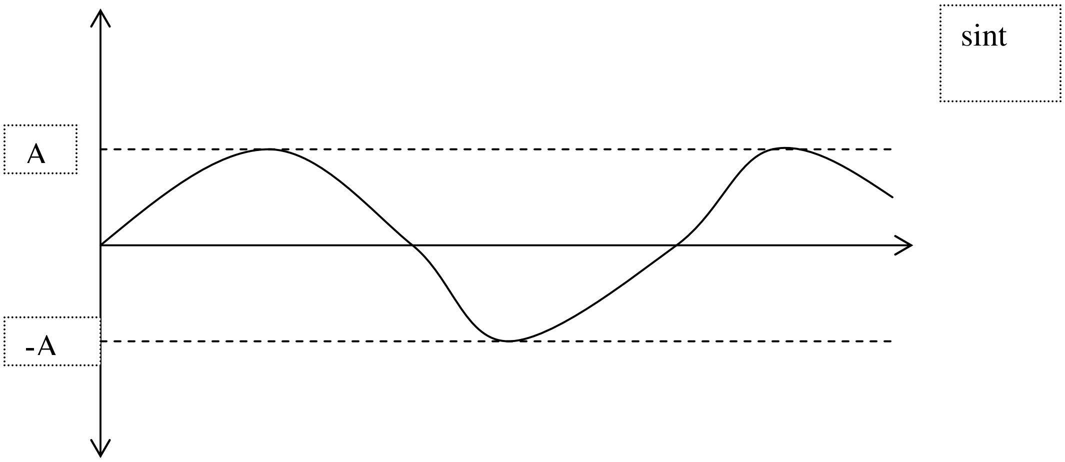
- **Remark3** : Consider the following simple example: $a_n = 0$ for all n, and $b_n = 0$ for all $n > 1$ (i.e., $n = 2, 3, 4, \ldots$) and $b_1 = b_2 = 1$, and $p = 2\pi$. Hence: $f(t) = \sin t + \sin 2t$

Also, according to the sum-product rule: $\sin t+\sin 2t=2\cos\left(\tfrac{1}{2}t\right)\sin\left(\tfrac{3}{2}t\right)=f(t)$

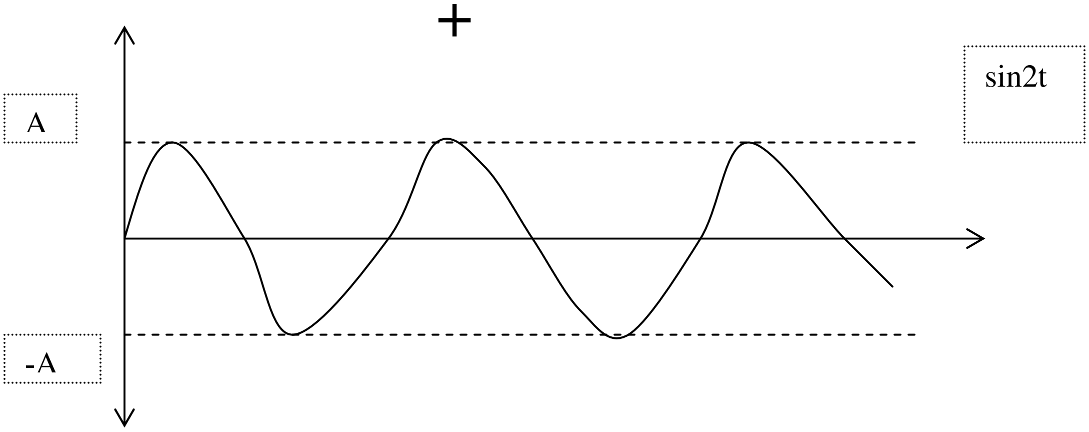
Hence $f(t)$ is the signal with carrier wave (envelope): $2\cos\left(\tfrac{1}{2}t\right)$ and phase: $\sin\left(\tfrac{3}{2}t\right)$. This signal is Fourier analyzed into the two linearly independent components $\sin t$ and $\sin 2t$. (Or read in reverse, superposing two linearly independent terms produces the signal: $\sin t+\sin 2t=2\cos\left(\tfrac{1}{2}t\right)\sin\left(\tfrac{3}{2}t\right)=f(t)$:


2A
-2A
$f(t)$

||


sint
A
-A

+


sin2t
A
-A

- **Remark4**: Recall that a function $f(t)$ is *even* provided that: $f(-t) = f(t)$ and *odd*, provided that: $f(-t) = -f(t)$. Any function can be decomposed into a sum of even and odd term:

  $$f(t) = evn_f(t) + odd_f(t) = \tfrac{1}{2}\left(f(t) + f(-t)\right) + \tfrac{1}{2}\left(f(t) - f(-t)\right)$$

  Given the Fourier series of $f(t)$:

  $$f(t) = \sum_{n=0}^{\infty}\left\{a_n \cos\left(\tfrac{2n\pi t}{p}\right) + b_n \sin\left(\tfrac{2n\pi t}{p}\right)\right\} = \sum_{n=0}^{\infty} a_n \cos\left(\tfrac{2n\pi t}{p}\right) + \sum_{n=0}^{\infty} b_n \sin\left(\tfrac{2n\pi t}{p}\right)$$

  Equating terms: $evn_f(t) = \sum_{n=0}^{\infty} a_n \cos\left(\tfrac{2n\pi t}{p}\right), odd_f(t) = \sum_{n=0}^{\infty} b_n \sin\left(\tfrac{2\pi nt}{p}\right)$

  (Hardly surprising, since cosine and sine are even and odd functions)

  1.) If $f(t) = f(-t)$, then $odd_f(t) = 0$ and hence $b_n = 0$ for all n.
  2.) If $f(t) = -f(-t)$, then $evn_f(t) = 0$ and hence $a_n = 0$ for all n.

- **Example**: Find the Fourier Series for $f(t) = t^2, 0 < t < 2\pi, p = 2\pi$

Keep in mind that this function is *not* even, since it consist of half-parabolae tiling the x-axis in units of 2π (defined in intervals: $I_k = [2k\pi, 2(k+1)\pi]$). In anticipaotion of the integration, it turns out to be evaluable by parts in such a manner amenable to Tabular Integration (See Handout 1a), since $u(t) = t^2$ has derivatives that converge to zero while the antiderivatives of $dv = \sin(nt)$, $\cos(nt)$ don't grow in complexity

Aside: Evaluating $\int t^2 \cos(nt)dt, \int t^2 \sin(nt)dt$:

u	***dv***
t^2 +	cos(nt)
$2t$ -	${}^1/_n \sin(nt)$
2 +	$-{}^1/_{n^2} \cos(nt)$
0	$-{}^1/_{n^3} \sin(nt)$

So: $\int t^2 \cos(nt)dt = \frac{t^2}{n}\sin nt + \frac{2t}{n^2}\cos nt - \frac{2}{n^3}\sin nt$

u	***dv***
t^2 +	sin(nt)
$2t$ -	$-{}^1\!/_n\cos(nt)$
2 +	$-{}^1\!/_{n^2}\sin(nt)$
0	${}^1\!/_{n^3}\cos(nt)$

So: $\int t^2 \sin(nt)dt = -\frac{t^2}{n}\cos nt + \frac{2t}{n^2}\sin nt + \frac{2}{n^3}\cos nt$

So, for all $n > 0$:

$$a_n = \frac{2}{2\pi}\int_0^p f(t)\cos\left(\frac{2n\pi t}{2\pi}\right)dt = \frac{1}{\pi}\int_0^{2\pi} t^2 \cos nt dt = \frac{1}{\pi}\left\{\frac{t^2}{n}\sin nt\Big|_0^{2\pi} + \frac{2t}{n^2}\cos nt\Big|_0^{2\pi} - \frac{2}{n^3}\sin nt\Big|_0^{2\pi}\right\}$$
$$= \frac{1}{\pi}\left\{0 + \frac{2}{n^2}\left(2\pi\cos n\pi - (0)\cos n0\right) - 0\right\} = \frac{4}{n^2}$$

$$b_n = \frac{2}{2\pi}\int_0^p f(t)\sin\left(\frac{2n\pi t}{2\pi}\right)dt = \frac{1}{\pi}\int_0^{2\pi} t^2 \sin nt dt = \frac{1}{\pi}\left\{-\frac{t^2}{n}\cos nt\Big|_0^{2\pi} + \frac{2t}{n^2}\sin nt\Big|_0^{2\pi} + \frac{2}{n^3}\cos nt\Big|_0^{2\pi}\right\}$$
$$= \frac{1}{\pi}\left\{-\frac{1}{n}\left(4\pi^2\cos 2n\pi - 0\cdot\cos 0\right) + 0 + \frac{2}{n^3}\left(\cos 2n\pi - \cos 0\right)\right\} = -\frac{4\pi}{n}$$

Now in the case of $n = 0$, b_0 is *always* = 0 (due to the sin(0) coefficient in the integral formula) however, a_0 may be non-zero:

$$a_0 = \frac{1}{2\pi}\int_0^{2\pi} t^2\cos(0t)dt = \frac{1}{\pi}\int_0^{2\pi} t^2 dt = \frac{1}{2\pi}\cdot\frac{1}{3}t^3\Big|_0^{2\pi} = \frac{4}{3}\pi^2$$

Hence:

$$f(t) = \sum_{n=0}^{\infty}\left\{a_n\cos\left(\frac{2n\pi t}{p}\right) + b_n\sin\left(\frac{2n\pi t}{p}\right)\right\} = a_0 + \sum_{n=1}^{\infty}\left[a_n\cos nt + b_n\sin nt\right]$$
$$= \frac{4}{3}\pi^2 + \sum_{n=1}^{\infty}\left[\frac{4}{n^2}\cos nt - \frac{4\pi}{n}\sin nt\right] = 4\left\{\frac{1}{3}\pi^2 + \sum_{n=1}^{\infty}\frac{1}{n}\left(\frac{1}{n}\cos nt - \pi\sin nt\right)\right\}$$