MA360 **Handout 9a (week of March 20)**
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PreExam II Exercises

I. a) Use THM12 to find the inverse LT of: $\frac{s^2}{(s^2+1)(s^2+9)}$

$$\frac{s^2}{(s^2+1)(s^2+3^2)} = \frac{s}{(s^2+1)}\cdot\frac{s}{(s^2+3^2)} = L\{\cos t\}L\{\cos 3t\} = L\{(\cos t * \cos 3t)\}$$

where:

$$\cos t * \cos 3t = \int_0^t \cos(t-u)\cos 3u\,du = \int_0^t \cos u \cos 3(t-u)\,du$$

$$= \tfrac{1}{2}\int_0^t [\cos(t-4u)+\cos(t+2u)]du = \tfrac{1}{2}\left\{-\tfrac{1}{4}\sin(t-4u)\Big|_0^t + \tfrac{1}{2}\sin(t+2u)\Big|_0^t\right\}$$

$$= -\tfrac{1}{8}\{-\sin 3t - \sin t\} + \tfrac{1}{4}(\sin 3t - \sin t) = \tfrac{3}{8}\sin 3t - \tfrac{1}{8}\sin t = \tfrac{1}{8}(3\sin 3t - \sin t)$$

b.) Use your result obtained in a) to solve: $\dot{x}(t) + \int_0^t x(\omega) = \cos 3t, x(0) = 0$

$$L\{\dot{x}(t) + \int_0^t x(\omega)d\omega\} = L\{\cos 3t\} \Rightarrow sY(s) - x(0) + \tfrac{1}{s}Y(s) = \frac{s}{s^2+3^2}$$

$$\Rightarrow \left(s+\tfrac{1}{s}\right)Y(s) = \frac{s}{s^2+9} \Rightarrow (s^2+1)Y(s) = \frac{s^2}{(s^2+9)} \Rightarrow Y(s) = \frac{s^2}{(s^2+1)(s^2+9)}$$

$$\therefore x(t) = L^{-1}\left\{\frac{s^2}{(s^2+1)(s^2+6)}\right\} = (from\_Ia) = \tfrac{1}{8}(3\sin 3t - \sin t)$$

II) Solve: $\dot{x} + 2x + 5\int_0^t x(\omega)d\omega = 4\delta(t-1)$, $x(0) = 0$

$$L\{\dot{x} + 2x + 5\int_0^t x(\omega)d\omega\} = L\{4\delta(t-1)\} \Rightarrow sY(s) + 2Y(s) + \tfrac{5}{s}Y(s) = 4e^{-s}$$

$$\Rightarrow (s^2+2s+5)Y(s) = 4se^{-s} \Rightarrow Y(s) = 4\frac{se^{-s}}{(s^2+2s+1+4)} = 4\frac{se^{-s}}{[(s+1)^2+2^2]}$$

<u>Method 1: Use THM7 & THM8:</u>

$$Y(s) = 4e^{-s}\frac{(s+1)-1}{[(s+1)^2+2^2]} = 4e^{-s}\left\{\frac{(s+1)}{[(s+1)^2+2^2]} - \frac{1}{[(s+1)^2+2^2]}\right\} = 4e^{-s}\{F(s+1) - G(s+1)\}$$

Where, according to THM7:

$F(s+1) = L\{e^{-t}f(t)\}$, where $F(s) = \frac{s}{s^2+2^2} = L\{f(t)\} \Rightarrow f(t) = \cos 2t$

$G(s+1) = L\{e^{-t}g(t)\}$, where $G(s) = \frac{1}{2}\frac{2}{s^2+2^2} = L\{g(t)\} \Rightarrow g(t) = \frac{1}{2}\sin 2t$

Hence, according to THM8:

$$Y(s) = 4e^{-s}\left\{L\left\{e^{-t}(\cos 2t - \tfrac{1}{2}\sin 2t)\right\}\right\} = 4L\left\{u(t-1)e^{-(t-1)}\left[\cos 2(t-1) - \tfrac{1}{2}\sin 2(t-1)\right]\right\}$$

So:

$$x(t) = 4u(t-1)e^{(1-t)}[\cos(2t-2) - \tfrac{1}{2}\sin(2t-2)] = 4ee^{-t}\left[\cos 2(t-1) - \tfrac{1}{2}\sin 2(t-1)\right], t > 1$$

<u>Method 2: Using Lemma1 (Handout 5b) & THM8:</u>

$$Y(s) = 4e^{-s}\,\frac{s}{\left[(s+1)^2+2^2\right]} \Rightarrow L^{-1}\left\{\frac{s+a}{(s+b)^2+\omega^2}\right\} = \frac{k}{\omega}e^{-bt}\sin(\omega t+\phi)$$

(where: $k = \sqrt{(a-b)^2+\omega^2}, \phi = \arctan\left(\frac{\omega}{a-b}\right)$)

Here: $a = 0$, $b = 1$, $\omega = 2$, $\phi = \tan^{-1}\left(\frac{2}{-1}\right) = \tan^{-1}(-2)$, $k = \sqrt{(0-1)^2+2^2} = \sqrt{5}$

Note: By the Pythagorean Theorem:

$\tan\phi = -2 = \frac{2}{-1} = \frac{O}{A} \Rightarrow H = \sqrt{5}$

$\Rightarrow \sin\phi = \sin(\arctan(-2)) = \frac{O}{H} = \frac{2}{\sqrt{5}}, \cos\phi = \frac{O}{H} = -\frac{1}{\sqrt{5}}$

Hence:

$$L^{-1}\left\{\frac{s}{(s+1)^2+2^2}\right\} = \frac{\sqrt{5}}{2}e^{-t}\sin(2t+\phi) = \frac{\sqrt{5}}{2}e^{-t}(\sin 2t\cos\phi + \cos 2t\sin\phi)$$

$$= \frac{\sqrt{5}}{2}e^{-t}\left(-\frac{1}{\sqrt{5}}\sin 2t + \frac{2}{\sqrt{5}}\cos 2t\right) = e^{-t}\left(\cos 2t - \tfrac{1}{2}\sin 2t\right)$$

Hence, according to THM8:

$$Y(s) = 4e^{-s}\left\{L\left\{e^{-t}(\cos 2t - \tfrac{1}{2}\sin 2t)\right\}\right\} = 4L\left\{u(t-1)e^{-(t-1)}\left[\cos 2(t-1) - \tfrac{1}{2}\sin 2(t-1)\right]\right\}$$

So: $x(t) = 4u(t-1)e^{(1-t)}[\cos(2t-2) - \tfrac{1}{2}\sin(2t-2)] = 4ee^{-t}\left[\cos 2(t-1) - \tfrac{1}{2}\sin 2(t-1)\right], t > 1$

III.) Solve: $\ddot{x} - 3\dot{x} + 2x = 2e^{-t}, x(0) = 2, \dot{x}(0) = -1$

(a) By the UC (Undetermined Coefficients) Method

Auxiliary Eqn.: $r^2 - 3r + 2 = (r-2)(r-1) = 0 \Rightarrow x_{tr}(t) = c_1e^{2t} + c_2e^{t}$

$g(t) = 2e^{-t} \Rightarrow x_{ss}(t) = Ae^{-t}, \dot{x}_{ss}(t) = -Ae^{-t}, \ddot{x}_{ss}(t) = x_{ss}(t) = Ae^{-t}$

$\ddot{x}_{ss}(t) - 3\dot{x}_{ss}(t) + 2x_{ss}(t) = 6Ae^{-t} = 2e^{-t} \Rightarrow A = \frac{1}{3}$

So:

$$x(t) = x_{tr}(t) + x_{ss}(t) = c_1 e^{2t} + c_2 e^{t} + \tfrac{1}{3} e^{-t}$$
$$x(0) = 2 = c_1 + c_2 + \tfrac{1}{3} \Rightarrow c_1 + c_2 = \tfrac{5}{3}$$
$$\dot{x}(t) = 2c_1 e^{2t} + c_2 e^{t} - \tfrac{1}{3} e^{-t}$$
$$\dot{x}(0) = -1 = 2c_1 + c_2 - \tfrac{1}{3} \Rightarrow 2c_1 + c_2 = -\tfrac{2}{3}$$
$$\therefore c_1 = -\tfrac{7}{3} \Rightarrow c_2 = 4$$

Hence: $x(t) = -\tfrac{7}{3} e^{2t} + 4e^{t} + \tfrac{1}{3} e^{-t}$

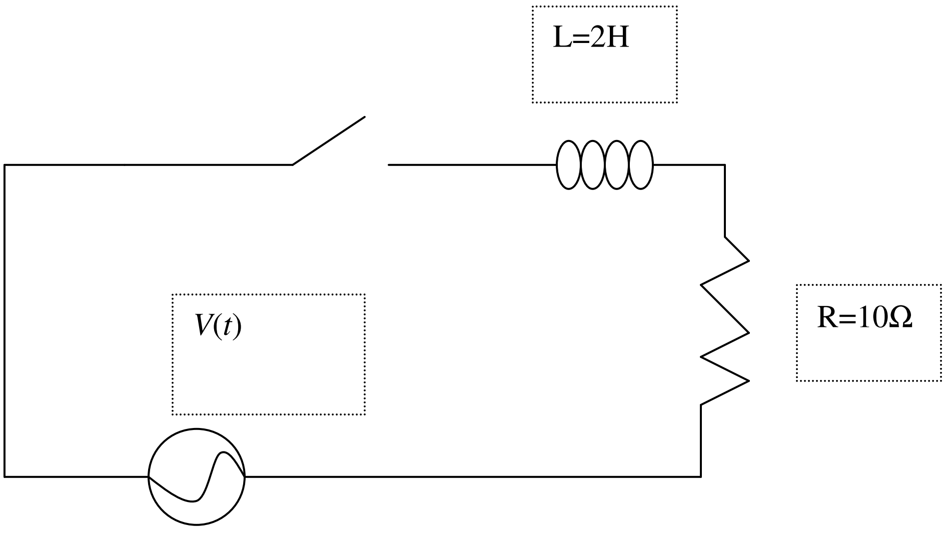
b) By the LT method

$$L\{\ddot{x} - 3\dot{x} + 2x\} = L\{2e^{-t}\} \Rightarrow \left(s^2 Y(s) - 2s + 1\right) - 3\left(sY(s) - 2\right) + 2Y(s) = \tfrac{2}{s+1}$$
$$\Rightarrow \left(s^2 - 3s + 2\right)Y(s) = \tfrac{2}{s+1} - 7 + 2s \Rightarrow (s-2)(s-1)Y(s) = \tfrac{2s^2 - 5s - 5}{s+1}$$
$$\Rightarrow Y(s) = \tfrac{2s^2 - 5(s+1)}{(s+1)(s-2)(s-1)} = \tfrac{A}{(s+1)} + \tfrac{B}{(s-2)} + \tfrac{C}{(s-1)}$$

By Heaviside Cover Method:

$$(s = -1) \Rightarrow A = \tfrac{2}{6} = \tfrac{1}{3}$$
$$(s = 2) \Rightarrow B = \tfrac{-7}{3 \cdot 1} = -\tfrac{7}{3}$$
$$(s = 1) \Rightarrow C = \tfrac{-8}{2 \cdot (-1)} = 4$$
$$\therefore Y(s) = \tfrac{1/3}{(s+1)} - \tfrac{7/3}{(s-2)} + \tfrac{4}{(s-1)} \Rightarrow x(t) = L^{-1}\{Y(s)\} = \tfrac{1}{3} e^{-t} - \tfrac{7}{3} e^{2t} + 4e^{t}$$

IV) A resistor of 10ohms is hooked up to a voltage source *V*(*t*) is series with an inductor of 2 henries, and a switch. Initially the current = 0. Find the current when:



Using Loop Rule: $V(t)=L\frac{di}{dt}+Ri \Rightarrow V(s)=LsI(s)+RI(s)\Rightarrow V(s)=2(s+5)I(s)$

a) V(t) = 20e^{-3t}

$$V(s)=L\{20e^{-3t}\}=\tfrac{20}{s+3}=2(s+5)I(s)\Rightarrow \tfrac{10}{s+3}=(s+5)I(s)$$
$$\Rightarrow I(s)=\tfrac{10}{(s+3)(s+5)}=\tfrac{A}{s+3}+\tfrac{B}{s+5}$$
$$\Rightarrow (s=-3)\Rightarrow A=\tfrac{10}{2}=5,(s=-5)\Rightarrow B=\tfrac{10}{-2}=-5$$
$$\therefore I(s)=\tfrac{5}{s+3}-\tfrac{5}{s+5}\Rightarrow i(t)=L^{-1}\{I(s)\}=5(e^{-3t}-e^{-5t})$$

b) $V(t)=50\sin 3t$

$$V(s)=L\{50\sin 3t\}=\tfrac{50}{s^2+3^2}=2(s+5)I(s)\Rightarrow \tfrac{50}{s^2+3^2}=(s+5)I(s)$$
$$\Rightarrow I(s)=\tfrac{50}{(s^2+3^2)(s+5)}$$

<u>Method 1 (THM12)</u>:

$$I(s)=50\tfrac{1}{s^2+3^2}\cdot\tfrac{1}{s+5}=50L\{\tfrac{1}{3}\sin 3t\}L\{e^{-5t}\}=\tfrac{50}{3}L\{(\sin 3t * e^{-5t})\}$$

where: $(\sin 3t * e^{-5t})=\int_0^t \sin 3(t-u)e^{-5u}\,du=\int_0^t \sin 3u e^{-5(t-u)}\,du$

(For algebraic simplicity, select the representation of the convolution on the right)

$$\int_0^t \sin 3u e^{-5(t-u)}\,du=e^{-5t}\int_0^t \sin 3u e^{5u}\,du=e^{-5t}\left\{\tfrac{e^{5u}}{5^2+3^2}(5\sin 3u-3\cos 3u)\Big|_0^t\right\}$$
$$=e^{-5t}\left\{\tfrac{e^{5t}}{34}(5\sin 3t-3\cos 3t)-\tfrac{1}{34}(-3)\right\}=\tfrac{e^{-5t}}{34}\left\{e^{5t}(5\sin 3t-3\cos 3t)+3\right\}$$
$$=\tfrac{1}{34}\left[(5\sin 3t-3\cos 3t)+3e^{-5t}\right]$$

Hence: $i(t)=\frac{50}{3}\cdot\frac{1}{34}\left[(5\sin 3t-3\cos 3t)+3e^{-5t}\right]=\frac{25}{51}\left[(5\sin 3t-3\cos t)+3e^{-5t}\right]$

<u>Method 2 (Partial Fractions):</u>

$$I(s)=\tfrac{50}{(s^2+9)(s+5)}=\tfrac{A_1s+A_2}{(s^2+9)}+\tfrac{B}{s+5}\Rightarrow 50=(A_1s+A_2)(s+5)+B(s^2+9)$$
$$(s=-5)\Rightarrow 50=34B\Rightarrow B=\tfrac{50}{34}=\tfrac{25}{17}$$
$$\Rightarrow 50=(A_1s+A_2)(s+5)+\tfrac{25}{17}(s^2+9)$$

$$(s=0) \Rightarrow 50 = 5A_2 + \tfrac{25\cdot 9}{17} \Rightarrow 5A_2 = 50 - \tfrac{225}{17} \Rightarrow 5A_2 = \tfrac{625}{17} \Rightarrow A_2 = \tfrac{125}{17}$$

Differentiating the above eqn.:

$$\begin{aligned}
&0 = \left(A_1 s + \tfrac{125}{17}\right) + A_1(s+5) + \tfrac{50}{17}s = \left(2A_1 + \tfrac{50}{17}\right)s + \left(5A_1 + \tfrac{125}{17}\right)\\
&s^1 : 2A_1 + \tfrac{50}{17} = 0 \Rightarrow A_1 = -\tfrac{25}{17}\\
&check : s^0 : 5A_1 + \tfrac{125}{17} = 0 = 5\cdot\left(-\tfrac{25}{17}\right) + \tfrac{125}{17}\, OK
\end{aligned}$$

Hence:

$$\begin{aligned}
&I(s) = \frac{-\frac{25}{17}s + \frac{125}{17}}{s^2+9} + \frac{\frac{25}{17}}{s+5} = \frac{1}{17}\left\{\frac{-25s+125}{s^2+9} + \frac{25}{s+5}\right\} = \frac{25}{17}\left\{\frac{-s+5}{s^2+9} + \frac{1}{s+5}\right\}\\
&\Rightarrow i(t) = \tfrac{25}{17}\left\{L^{-1}\left\{\tfrac{-s+5}{s^2+3^2}\right\} + L^{-1}\left\{\tfrac{1}{s+5}\right\}\right\}
\end{aligned}$$

Using Lemma1 (Handout 5b) on the first term:

$$L^{-1}\left\{\frac{s+a}{(s+b)^2+\omega^2}\right\} = \frac{k}{\omega}e^{-bt}\sin(\omega t + \phi) \qquad \text{(where: } k = \sqrt{(a-b)^2 + \omega^2}, \phi = \arctan\left(\tfrac{\omega}{a-b}\right)\text{)}$$

Hence: $L^{-1}\left\{\frac{-s+5}{(s+0)^2+3^2}\right\} = -L^{-1}\left\{\frac{s-5}{(s-0)^2+3^2}\right\} = -\frac{\sqrt{34}}{3}e^{0t}\sin(3t+\phi)$

Where:

$$\begin{aligned}
&\phi = \tan^{-1}\left(\tfrac{3}{-5}\right) \Rightarrow \tan\phi = \tfrac{3}{-5} \Rightarrow O = 3, A = -5 \Rightarrow H = \sqrt{A^2 + O^2} = \sqrt{34}\\
&\therefore \cos\phi = \tfrac{A}{H} = -\tfrac{5}{\sqrt{34}}, \sin\phi = \tfrac{O}{H} = \tfrac{3}{\sqrt{34}}
\end{aligned}$$

So:

$$\begin{aligned}
&L^{-1}\left\{\frac{-s+5}{(s+0)^2+3^2}\right\} = -L^{-1}\left\{\frac{s-5}{(s-0)^2+3^2}\right\} = -\frac{\sqrt{34}}{3}(\sin 3t\cos\phi + \cos 3t\sin\phi)\\
&= -\frac{\sqrt{34}}{3}\left(-\frac{5}{\sqrt{34}}\sin 3t + \frac{3}{\sqrt{34}}\cos 3t\right) = \frac{5}{3}\sin 3t - \cos 3t
\end{aligned}$$

Hence:

$$\begin{aligned}
&i(t) = \tfrac{25}{17}\left\{L^{-1}\left\{\tfrac{-s+5}{s^2+3^2}\right\} + L^{-1}\left\{\tfrac{1}{s+5}\right\}\right\} = \tfrac{25}{17}\left\{\tfrac{5}{3}\sin 3t - \cos 3t + e^{-5t}\right\}\\
&= \tfrac{25}{51}\left[(5\sin 3t - 3\cos 3t) + 3e^{-5t}\right]
\end{aligned}$$