MA360 **Handout 8b (week of March 13)**
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Assignment: Aside from finishing Homework II, work problems 3,5,7 (pp 98-99, Sheng)

Applications: AC Circuits

In the case of circuits, the geometry of the situation is so restricted that the ordinarily rather intricate expressions of Maxwell's EM field equations[1] assume simple form. For example, Gauss's Law (macroscopic-integral version) becomes:

$$V_C = \tfrac{1}{C}Q(t) \Rightarrow V_C = \tfrac{1}{C}\int_0^t i(\omega)d\omega$$

(where: $i(t)$ is the current of the system, and C is the capacitance of the system, a constant determined by the electric permitivity and the geometry of the capacitor, i.e. the system holding the charge.)

Similarly, Ohm's Law (macroscopic-integral version) becomes:

$$V_R = Ri(t)$$

(where: R is the macroscopic resistance of the system, expressed in terms of the microscopic *resistivity* ρ according to: $R = {}^{\rho L}/_{A}$, where: A is the cross-sectional area of the conductor and L is its unit length.)

Faraday's Law, for cylindrically symmetric inductors (coils) becomes:

$$V_L(t) = L\tfrac{d}{dt}i(t)$$

[1] Which are: 1) Gauss's Law, 2) No magnetic monopole law, 3) Ampere's Law, 4) Faraday's Law (modified to include displacement current for time-varying currents.) In differential form, they comprise a set of four coupled nonlinear differential equations, such that (in free space, i.e. no sources of charge density or current density): 1) $\vec{\nabla}\cdot\vec{E} = \frac{\rho}{\varepsilon_0}$, 2) $\vec{\nabla}\cdot\vec{B} = 0$, 3) $\vec{\nabla}\times\vec{B} = \mu_0\varepsilon_0\frac{\partial}{\partial t}\vec{E}$,4) $\vec{\nabla}\times\vec{E} = -\frac{\partial}{\partial t}\vec{B}$ where $\vec{E},\vec{B}$ are the electric and magnetic fields, (In general, time-dependents vector-valued vector functions, i.e $R^3 \to R^3$ functions , where R are the real numbers.) Moreover, the quantities $\vec{\nabla}\cdot, \vec{\nabla}\times$, express two different kinds of vector derivatives (note: $\vec{\nabla} = \frac{\partial}{\partial x}\hat{i} + \frac{\partial}{\partial y}\hat{j} + \frac{\partial}{\partial z}\hat{k}$ is the 3D "del" operator, or vector-valued derivative operator), the scalar (dot product) or "divergence," and the vector (cross product) "curl." Physically speaking, a positive divergence signifies a voltage *source*, a negative divergence signifies a voltage *sink*, whereas a non-zero *curl* indicates a tendency for the field (whether magnetic or electric) to form a *vortex*. Conversely, a zero curl implies a *laminar* (or vortex-free) EM vector field.

(where: L is the macroscopic *inductance* of the system, which, depends on the geomrtry of the coil-apparatus and measures the (non-conservative) voltage produced inevitably by a time-varying magnetic field, inducing *eddy currents* across the coil)

Observe that the Laplace Transforms of the aforementioned are, according to Thm2 and Thm4:

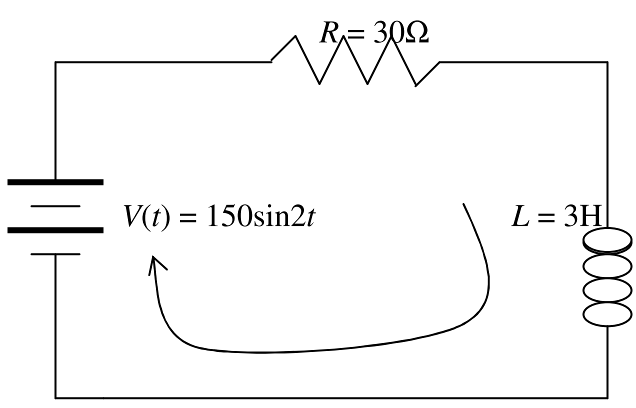
$$V_C(t) = \tfrac{1}{C}\int_0^t i(\omega)d\omega \Rightarrow V_C(s) = \tfrac{1}{Cs} I(s)$$
$$V_R(t) = Ri(t) \Rightarrow V_R(s) = RI(s)$$
$$V_L(t) = L\tfrac{d}{dt} i(t) = L(sI(s) - i(0))$$

Note: Perhaps you're familiar with Ohm's law generalized for AC currents for LRC circuits, which reads:

> $V = ZI$, where: $Z = \sqrt{(X_L - X_C)^2 + R^2}$ is the overall *impedance* of the circuit (measured in Ω, i.e. ohms) where: $X_L = \omega L = 2\pi f L, X_C = \frac{1}{\omega C} = \frac{1}{2\pi f C}$ are the overall *inductive* and *capacitive* reactances, likewise measured in ohms. Now you have an insight into the nature of these formulae: ω, (or angular frequency) *is the Laplace transform for the t domain!* For example, if the initial conditions = 0, observe: $V_L(t) = L\frac{d}{dt} i(t) = L(sI(s)) = (sL)I$, which is precisely the version of Ohm's law for the inductor alone, where: $V_L(s) = X_L I(s) \equiv (sL)I(s)$

Mathematically speaking, LTs enable us to transform AC circuit problems, normally involving second-order ODEs, into algebraic equations which are similar in form to the algebraic equations you're accustomed to solving for DC circuits.

- Example (Problem 2b, p98, Sheng) (assume zero initial conditions)



According to the Loop Rule: $V(t) = 150\sin 2t = Ri(t) + Li'(t)$

This first-order ODE, in s-domain, becomes the rather straightforward algebraic expression:

$$\frac{150\cdot 2}{s^2+4}=\frac{300}{s^2+4}=RI(s)+LsI(s)\Rightarrow I(s)=\frac{300}{(s^2+4)(R+sL)}=\frac{300}{(s^2+4)(30+3s)}=\frac{100}{(s^2+4)(s+10)}$$

…which, in particular, can be solved either by partial fractions or by THM12

Method1: (Partial fractions)

$$\frac{100}{(s^2+4)(s+10)}=\frac{A_1s+A_2}{s^2+4}+\frac{B}{s+10}\Rightarrow 100=(A_1s+A_2)(s+10)+B(s^2+4)$$
$$(s=-10)\Rightarrow 100=104B\Rightarrow B=\frac{100}{104}=\frac{25}{26}$$
$$(s=0)\Rightarrow 100=10A_2+\frac{50}{13}\Rightarrow 10=A_2+\frac{5}{13}\Rightarrow A_2=\frac{125}{13}$$
$$(s=1)\Rightarrow 100=11\left(A_1+\frac{125}{13}\right)+\frac{125}{26}\Rightarrow A_1+\frac{125}{13}=\frac{225}{26}\Rightarrow A_1=-\frac{25}{26}$$

So: $I(s)=\frac{100}{(s^2+4)(s+10)}=\frac{-(25/26)s+125/13}{s^2+4}+\frac{25/26}{s+10}=25\left\{\frac{-s/26}{s^2+2^2}+\frac{5/13}{s^2+2^2}+\frac{1/26}{s+10}\right\}$

Hence, the inverse LT of the first term can be obtained using Lemma2, as well as be straightforwardly obtained based on the way the fraction has been split up

$$I(s)=25\left\{\frac{-s/26}{s^2+2^2}+\frac{5/13}{s^2+2^2}+\frac{5/26}{s+10}\right\}=25\left\{-\frac{1}{26}L\{\cos 2t\}+\frac{5}{26}L\{\sin 2t\}+\frac{1}{26}L\{e^{-10t}\}\right\}$$
$$\Rightarrow i(t)=-\frac{25}{26}\cos 2t+\frac{125}{26}\sin 2t+\frac{25}{26}e^{-10t}$$

Method2: THM12

$$\frac{100}{(s^2+4)(s+10)}=100\left\{\frac{1}{s^2+4}\cdot\frac{1}{s+10}\right\}=100\left(L\{\tfrac{1}{2}\sin 2t\}L\{e^{-10t}\}\right)=50L\{(\sin 2t*e^{-10t})\}$$

Where: $\sin 2t*e^{-10t}=\int_0^t \sin 2(t-u)e^{-10u}du=\int_0^t \sin 2u e^{-10(t-u)}du$

The second form of the convolution integral requires fewer steps to evaluate. Hence:

$$\sin 2t*e^{-10t}=\int_0^t \sin 2u e^{-10(t-u)}du=e^{-10t}\int_0^t e^{10u}\sin 2u\,du$$
$$=e^{-10t}\left\{\frac{e^{10u}}{10^2+2^2}(10\sin 2u-2\cos 2u)\Big|_0^t\right\}=\frac{e^{-10t}}{104}\left\{e^{10t}(10\sin 2t-2\cos 2t)+2\right\}$$
$$=\frac{e^{-10t}}{52}\left\{e^{10t}(5\sin 2t-\cos 2t)+1\right\}=\frac{1}{52}\left(5\sin 2t-\cos 2t+e^{-10t}\right)$$

(using Formula 30, A7 in the integration)

Hence: $i(t) = L^{-1}\left\{\frac{100}{(s^2+4)(s+10)}\right\} = 50\left[\left(\sin 2t * e^{-10t}\right)\right] = \frac{50}{52}\left(5\sin 2t - \cos 2t + e^{-10t}\right)$

$= \frac{25}{26}\left(5\sin 2t - \cos 2t + e^{-10t}\right) = \frac{125}{26}\sin 2t - \frac{25}{26}\cos 2t + \frac{25}{26}e^{-10t}$

HINTS Assignment II

I.a) You can use either the formula in Lemma2 (p6 Handout 5b) or alternatively you can split up the numerator term and complete the square on the denominator term as follows:

$$\frac{s^2+2s}{\left(s^2+2s+2\right)^2} = \frac{(s^2+2s+2)-2}{\left(s^2+2s+2\right)^2} = \frac{\left(s^2+2s+2\right)}{\left(s^2+2s+2\right)^2} - \frac{2}{\left(s^2+2s+2\right)} = \frac{1}{\left[(s+1)^2+1^2\right]} - \frac{2}{\left[(s+1)^2+1^2\right]^2}$$

And use THM7 on the first fraction and THM12 on the second.

I.b) It's most efficient to use partial fractions. Keep in mind you MUST use THM7 to obtain the inverse LTs for all the terms on the right hand side.

IIa.) Most straightforward approach is of course the method of undetermined coefficients (UC). (You could apply the VP approach, i.e. the variations of parameters, but the algebra is messier)

IIb) Once you've obtained the LT of this equation, and after isolating $Y(s)$, you'll have two fractions (or three, if you've left the LT on the right hand side terms, i.e. the ($\sin 2t$ + $\cos 2t$) terms as separate fraction…hint: easier to combine as one single fraction!). It's most efficient to apply the technique of partial fractions on the first one, (reading left to right), expanded in terms of two quadratic irreducibles. (I would *not* recommend decomposing the two quadratic irreducibles into linear terms with complex-valued roots, the algebra is far messier!) The second fraction is easily resolved using THM7. The coefficient set, when you applied the method of partial fractions on the first term should be: $\left\{\frac{1}{5}, \frac{3}{10}, \frac{2}{5}, -\frac{3}{10}\right\}$ *if* you followed by suggestion regarding combining the LTs of ($\sin 2t$ + $\cos 2t$) into one single fraction!

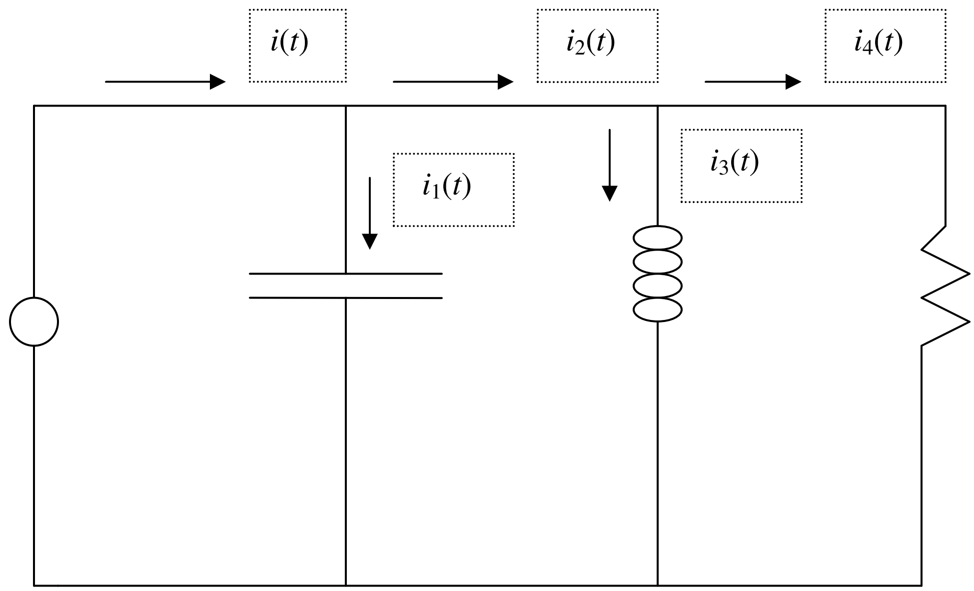
IIIa) Most stratightforwardly solvable if you use THM12. Upon isolating $Y(s)$ you'll have the following expression:

$$2\frac{s}{s^2+4}\cdot\frac{1}{s^2+1} = 2L\{\cos 2t\}L\{\sin t\} = 2L\{\cos 2t * \sin t\} = \text{etc}\ldots$$

See Handouts 7, 8b

IIIb) See Handout 7

IV) (a) You need to apply the junction rule twice. Your schematic should look like (though you need not label the currents in the same way):



Laplace tramsform the voltage of the AC source ($V(s)$), as well as the voltages across the capacitor, inductor, and resistor, ($V_C(s)$, $V_L(s)$, $V_R(s)$, respectively) according to the formulae derived in Handout 8b. Now according to the loop rule, you know immediately that: $V(s) = V_C(s)$, $V(s) = V_L(s)$, $V(s) = V_R(s)$. So including the junction rules, you have five equations involving the four unknowns $I_1(s)$, $I_2(s)$, $I_3(s)$, $I_4(s)$ (the Laplace Transformed currents). Basically you don't care about $I_2(s)$ so eliminate it by combining both junction rules (i.e., combine $I(s) = I_1(s) + I_2(s)$ and $I_2(s) = I_3(s) + I_4(s)$ by substituting for $I_2(s)$) Of course you *know* how to find the LT for $i(t)$ [see Handout 4b].

Now you can use $V(s) = V_L(s)$, $V(s) = V_C(s)$ to derive a simple equality for $I_1(s)$ and $I_3(s)$ (i.e., by eliminating $V(s)$, i.e. $V(s) = V_L(s)$ & $V(s) = V_C(s)$ obviously imply: $V_L(s) = V_C(s)$ By the same token, you can derive another simple equality for $I_3(s)$ and $I_4(s)$ through: $V_L(s) = V_R(s)$. Once you've done that, based on your junction rule above (in which you've already eliminated $I_2(s)$) you'll have a single equation for $I_1(s)$. Multiply this equation on both sides by s^2. Isolating $I_1(s)$, you should have the following expression:

$$I_1(s) = \frac{s(1-e^{-s})}{(s+1)^2}$$

Now it's easy to find the voltage across the capacitor, since: $V_C = \dfrac{I_1(s)}{Cs}$

To find the inverse LT for $V_C = \frac{I_1(s)}{Cs}$, it's easiest to split the above expression into two fractions. You'll apply THM7 on the first one and you'll apply BOTH THM7 and THM8 on the second terms.

IV) (b) Discussed in class

BONUS:

According to Newton's Second Law: NSL $\sum F = ma = m\ddot{x}(t)$, where:

$\sum F = -\beta\dot{x}(t) - \kappa x(t) + F(t)$ (the drag and compression forces get the negative sign since they act in opposition to the direction of the displacement of the bob.) $F(t)$ is of course the function expressing the driving force of a mechanism driving the oscillator.

1.) Express NSL in standard form (i.e. as a linear second order differential equation with constant coefficients.)
2.) Laplace transform this equation, and isolate $Y(s)$. You should get the following expression:

$$Y(s) = \frac{F(t)}{(ms^2 + \beta s + \kappa)}$$

3.) Complete the square on the denominator term. You'll get a rather messy third constant term, of the form: $\frac{\kappa}{m} - \frac{\beta^2}{4m^2}$. Go ahead and re-name it as: ω^2, (this is the *frequency* of the system), or something simple like that.
4.) Now three different cases can occur:

<u>Case1:</u> $\omega^2 > 0 \Rightarrow \kappa > \frac{\beta^2}{4m}$ (this is the *underdamped* case)
<u>Case2:</u> $\omega^2 = 0 \Rightarrow \kappa = \frac{\beta^2}{4m}$ (this is the *critically damped* case)
<u>Case3:</u> $\omega^2 < 0 \Rightarrow \kappa < \frac{\beta^2}{4m}$ (this is the *overdamped* case)

Set up appropriate fractions for Y(s) for the three separate cases and use THM12 to obtain formulae for $x(t)$ for each of these three cases (Your answers must obviously be left in integral form, since you don't know what $F(t)$ is.) <u>Case2:</u>
$\omega^2 = 0 \Rightarrow \kappa = \frac{\beta^2}{4m}$ (this is the *critically damped* case)