MA360 **Handout 7 (week of Feb 27)**
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<u>Assignment:</u> page 80 Sheng (Exercise 7) Problems 1,3,5,7 using Laplace Transform Method, and 2(a) (p86)

Solving linear ordinary differential and integro-differential equations (with constant coefficients—i.e., ODE & OIDE) using the LT method essentially comprises application of THM 2 (along with Corollary 2a) and THM4, to convert the equation into its appropriate algebraic counterpart in *s*-space. Thereafter, finding the inverse LT in order to solve the ODE and OIDE may involve usage of the other theorems, such as THMs 7,8,12, as well as the Partial Fraction Technique.

- **<u>Example (Problem 4 p80 Sheng)</u>**

$\ddot{x}(t)+16x(t)=5\sin t, x(0)=0=\dot{x}(0)$. The solution (Handout 6b, pp. 3 –6) was obtained using two methods (Undetermined Coefficients and the Variation of Parameters). Using the technique of LTs:

According to Corollary 2a:

$$L\{\ddot{x}(t)\}=s^2L\{x(t)\}-[sx(0)+\dot{x}(0)]=s^2L\{x(t)\}-[s\cdot 0-0]=s^2L\{x(t)\}$$

So applying the LT on both sides of the above equation:

$$s^2L\{x(t)\}+16L\{x(t)\}=L\{5\sin t\}=\tfrac{5}{s^2+1}$$

For simplicity, denote: $L\{x(t)\} = Y(s)$. Hence:

$$(s^2+16)Y(s)=\tfrac{5}{s^2+1}\Rightarrow Y(s)=5\cdot\tfrac{1}{s^2+16}\cdot\tfrac{1}{s^2+1}$$

Solving this ODE hence means finding the inverse LT of the above equation, i.e.:

$$x(t)=L^{-1}\{Y(s)\}=L^{-1}\left\{5\cdot\tfrac{1}{s^2+16}\cdot\tfrac{1}{s^2+1}\right\}=5L^{-1}\left\{\tfrac{1}{s^2+16}\cdot\tfrac{1}{s^2+1}\right\}$$

The above is most easily obtained using THM12:

$$\tfrac{1}{s^2+4^2}\cdot\tfrac{1}{s^2+1}=\left(\tfrac{1}{4}\cdot\tfrac{4}{s^2+4^2}\right)\cdot\left(\tfrac{1}{s^2+1}\right)=\left(\tfrac{1}{4}L\{\sin 4t\}\right)\cdot L\{\sin t\}=L\left\{\tfrac{1}{4}\int_0^t \sin 4u\sin(t-u)du\right\}$$

So: $L^{-1}\left\{\tfrac{1}{s^2+4^2}\cdot\tfrac{1}{s^2+1}\right\}=\tfrac{1}{4}\int_0^t \sin 4u\sin(t-u)du=\tfrac{1}{4}\int_0^t \tfrac{1}{2}[\cos(4u-t+u)-\cos(4u+t-u)]$

Note1: The Product-sum trig identity was used to simplify integral, in the last step above

$$= \tfrac{1}{8}\int_0^t [\cos(5u-t)-\cos(3u+t)]du = \tfrac{1}{8}\int_0^t \cos(5u-t)du - \tfrac{1}{8}\int_0^t \cos(3u+t)du$$
$$= \tfrac{1}{40}\sin(5u-t)\Big|_0^t - \tfrac{1}{24}\sin(3u+t)\Big|_0^t = \tfrac{1}{40}(\sin 4t - \sin(-t)) - \tfrac{1}{24}(\sin 4t - \sin t)$$
$$= \tfrac{1}{40}\sin 4t + \tfrac{1}{40}\sin t - \tfrac{1}{24}\sin 4t + \tfrac{1}{24}\sin t$$

Note2: The variable of integration is *u*. Then *t* behaves like a constant.

Hence:

$$x(t) = 5L^{-1}\left\{\tfrac{1}{s^2+16}\cdot\tfrac{1}{s^2+1}\right\} = 5\left\{\left(\tfrac{1}{40}-\tfrac{1}{24}\right)\sin 4t + \left(\tfrac{1}{40}+\tfrac{1}{24}\right)\sin t\right\}$$
$$= \left(\tfrac{1}{8}-\tfrac{5}{24}\right)\sin 4t + \left(\tfrac{1}{8}+\tfrac{5}{24}\right)\sin t = -\tfrac{1}{12}\sin 4t + \tfrac{1}{3}\sin t$$

- Example (Problem 6, p 80 Sheng)

$$\ddot{x}(t) + 4\dot{x}(t) + 8x(t) = 4\cos t + 7\sin t, x(0) = 1, \dot{x}(0) = -1$$

From Handout 6b , the solution $x(t) = e^{-2t}\cos 2t + \sin t$ was obtained using the UC method. Applying LTs:

$$L\{\ddot{x}(t)\} = s^2 L\{x(t)\} - [sx(0) + \dot{x}(0)] = s^2 L\{x(t)\} - [s\cdot 1 - 1] = s^2 Y - s + 1$$
$$L\{\dot{x}(t)\} = sL\{x(t)\} - x(0) = sY - 1$$

Hence:

$$L\{\ddot{x}\} + 4L\{\dot{x}\} + 8L\{x\} = (s^2 Y - s + 1) + 4(sY - 1) + 8Y = L\{4\cos t + 7\sin t\} = \tfrac{4s}{s^2+1} + \tfrac{7}{s^2+1}$$

So in *s*-space, the ODE becomes:

$$(s^2 + 4s + 8)Y - s + 1 - 4 = \tfrac{4s+7}{s^2+1} \Rightarrow Y(s) = \tfrac{4s+7}{(s^2+4s+8)(s^2+1)} + \tfrac{s+3}{(s^2+4s+8)}$$

(a) Using Partial Fractions on the first term:

$$\tfrac{4s+7}{(s^2+4s+8)(s^2+1)} = \tfrac{As+B}{(s^2+4s+8)} + \tfrac{Cs+D}{(s^2+1)} \Rightarrow 4s+7 = (As+B)(s^2+1) + (Cs+D)(s^2+4s+8)$$
$$s^3 : 0 = A + C \Rightarrow A = -C$$
$$s^2 : 0 = B + D + 4C = B + D - 4A$$
$$s^1 : 4 = A + 4D + 8C = A + 4D - 8A = -7A + 4D$$
$$s^0 : 7 = B + 8D$$

$$s^0 - s^2 : 7 = 7D + 4A \Rightarrow 28 = 28D + 16A$$
$$s^1 : 4 = -7A + 4D \Rightarrow -28 = 49A - 28D$$
$$\Rightarrow 0 = 65A \Rightarrow A = 0, D = 1 \Rightarrow C = 0, B = -1$$

Hence:

$$\tfrac{4s+7}{(s^2+4s+8)(s^2+1)} = \tfrac{-1}{(s^2+4s+8)} + \tfrac{1}{(s^2+1)}$$

(Check: $\tfrac{4s+7}{(s^2+4s+8)(s^2+1)} = \tfrac{-1}{(s^2+4s+8)} + \tfrac{1}{(s^2+1)} = \tfrac{-s^2-1+s^2+4s+8}{(s^2+4s+8)(s^2+11)} = \tfrac{4s+7}{(s^2+4s+8)(s^2+1)}$)

So:

$$Y(s) = \tfrac{4s+7}{(s^2+4s+8)(s^2+1)} + \tfrac{s+3}{(s^2+4s+8)} = \tfrac{-1}{(s^2+4s+8)} + \tfrac{1}{(s^2+1)} + \tfrac{s+3}{(s^2+4s+8)} = \tfrac{s+2}{(s^2+4s+8)} + \tfrac{1}{(s^2+1)}$$

a.) The inverse LT of the first term can be obtained via the formula in Lemma1 (Handout 5b)

$\tfrac{s+2}{(s^2+4s+8)} = \tfrac{s+2}{[(s+2)^2+2^2]} \Rightarrow a = b = \omega = 2,$ and

$$k = \sqrt{(2-2)^2 + 2^2} = 2, \phi = \arctan\left(\tfrac{2}{2-2}\right) = \arctan(\infty) = \tfrac{\pi}{2}$$

So: $L^{-1}\left\{\tfrac{s+2}{(s+2)^2+2^2}\right\} = \tfrac{2}{2} e^{-2t} \sin(2t + \phi) = e^{-2t} \sin\left(2t + \tfrac{\pi}{2}\right) = e^{-2t} \cos 2t$

b.) And of course the inverse LT for the second term $\tfrac{1}{s^2+1}$ is just: $\sin t$

Hence: $x(t) = L^{-1}\{Y(s)\} = e^{-2t} \cos 2t + \sin t$

It's important to keep in mind that the LT method is more general (and hence more powerful) than the other methods you're familiar with. For instance, (a) Integro-differential equations can be easily solved as well as higher-order linear ODEs with constant coefficients. Two examples are given below:

- Example (Sheng, 2(c), p86)

$$\dot{x}(t) + 2x + 2\int_0^t x(\omega)d\omega = 50, x(0) = 0$$

Applying the LT to both sides (using THM2, THM4):

$$L\{\dot{x}\}+2L\{x\}+2L\left\{\int_0^t x(\omega)d\omega\right\}=sL\{x(t)\}-x(0)+2L\{x(t)\}+\tfrac{2}{s}L\{x(t)\}=50L\{t^0\}$$
$$\Rightarrow sY(s)+2Y(s)+\tfrac{2}{s}Y(s)=\tfrac{50}{s}\Rightarrow s^2Y(s)+2sY(s)+2Y(s)=50\Rightarrow (s^2+2s+2)Y(s)=50$$
$$\Rightarrow Y(s)=\tfrac{50}{(s^2+2s+2)}$$

So the final result is just a simple quadratic irreducible. One could apply the formula from Lemma1, but it's simple enough to be amenable to a more 'honest' approach (in the sense that one isn't just invoking a formula mechanically) involving completing the square and applying shifting theorems:

$$\tfrac{50}{(s+1)^2+1^2}=50F(s+1)\Rightarrow F(s)=\tfrac{1}{s^2+(1)^2}=L\{\sin t\}\Rightarrow (THM7)\Rightarrow F(s+1)=L\{e^{-t}\sin t\}$$
$$\therefore x(t)=L^{-1}\{Y(s)\}=50F(s+\tfrac{1}{2})=50e^{-t}\sin t$$

- Example: (A 4$^{\text{th}}$ order linear ODE with constant coefficients)

$$x^{(4)}(t)+2\ddot{x}(t)+x(t)=\sin t$$
$$x(0)=1, \dot{x}(0)=-2, \ddot{x}(0)=3, x^{(3)}(0)=0$$

Applying Corollary2a:

$$L\{x^{(4)}(t)\}=s^4L\{x(t)\}-\left[s^3x(0)+s^2\dot{x}(0)+s\ddot{x}(0)+x^{(3)}(0)\right]=s^4Y(s)-s^3+2s^2-3s$$
$$L\{\ddot{x}(t)\}=s^2L\{x(t)\}-[sx(0)+\dot{x}(0)]=s^2Y(s)-s+2$$

So, applying the LT on both sides of the ODE:

$$(s^4Y-s^3+2s^2-3s)+2(s^2Y-s+2)+Y=\tfrac{1}{s^2+1}$$
$$(s^4+2s^2+1)Y+(-s^3+2s^2-5s+4)=\tfrac{1}{s^2+1}$$
$$(s^2+1)^2Y=\tfrac{1}{s^2+1}+s^3-2s^2+5s-4$$
$$\Rightarrow Y=\tfrac{1}{(s^2+1)^3}+\tfrac{s^3-2s^2+5s-4}{(s^2+1)^2}=\tfrac{1}{(s^2+1)^3}+\tfrac{(s^3+s)-2(s^2+1)+4s-2}{(s^2+1)^2}$$
$$=\tfrac{1}{(s^2+1)^3}+\tfrac{s(s^2+1)}{(s^2+1)^2}-\tfrac{2(s^2+1)}{(s^2+1)^2}+\tfrac{4s-2}{(s^2+1)}=\tfrac{1}{(s^2+1)^3}+\tfrac{s}{(s^2+1)}-\tfrac{2}{(s^2+1)}+\tfrac{4s-2}{(s^2+1)^2}$$

So after rearranging the numerator terms above, the final expression shows that the first and the fourth terms will require extra steps to obtain the inverse LTs (the second and third terms are of course the inverse LTs of cos*t* and –2sin*t*, respectively).

a.) To find the inverse LT of the first term, apply THM12. (It's fruitless to attempt Partial Fractions on it, since the numerator term is simple enough that you won't profit, i.e., you'll end up with the exact same expression if you go through and try to find the constants!)

$$\tfrac{1}{(s^2+1)^3} = \tfrac{1}{(s^2+1)} \cdot \tfrac{1}{(s^2+1)^2} = L\{f(t)\}L\{g(t)\}$$

Now certainly: $f(t) = \sin t$.

To obtain $g(t)$, we can use Lemma2 (Handout5b):

If: $F(s) = \dfrac{N(s)}{\left[(s-a)^2 + b^2\right]^2}$,

then: $L^{-1}\{F(s)\} = \tfrac{e^{at}}{2b^3}\left[(q - bp^* - bpt)\cos bt + (p + bq^* + bqt)\sin bt\right]$

where: $k = a + ib \Rightarrow R(s) = N(s), R(k) = p + iq, R'(k) = p^* + iq^*$

Here:

$$F(s) = \frac{1}{\left[(s-0)^2 + 1^2\right]^2} \Rightarrow a = 0, b = 1, k = a + bi = i,$$

$$R(s) = R(k) = 1 = p + iq \Rightarrow p = 1, q = 0,$$

$$R'(s) = 0 = R'(k) = p^* + iq^* \Rightarrow p^* = 0 = q^*$$

So: $L^{-1}\left\{\tfrac{1}{(s^2+1)^2}\right\} = g(t) = \tfrac{e^{0t}}{2}\left[(0 - 0 - t)\cos bt + (1 + 0 + 0)\sin bt\right] = \tfrac{1}{2}(-t\cos t + \sin t)$

Hence according to THM12: $\tfrac{1}{(s^2+1)^3} = L\{f(t)\}L\{g(t)\} = L\{(f * g)(t)\}$

Where:

$$(f * g)(t) = \int_0^t f(t-u)g(u)\,du = \int_0^t \sin(t-u)\left[\tfrac{1}{2}(-u\cos u + \sin u)\right]du$$

$$= \tfrac{1}{2}\int_0^t \sin(t-u)(-u\cos u + \sin u)\,du = \tfrac{1}{2}\int_0^t \left[-u\sin(t-u)\cos u + \sin(t-u)\sin u\right]du$$

Applying the trig product-sum identities on these two terms (in the last expression):

$$= \tfrac{1}{4}\int_0^t \left[-u(\sin t - \sin(2u - t)) + \cos(t - 2u) - \cos t\right]du$$

Splitting up this integral into four (note: t behaves like a constant term, since u is the dummy variable:

$$= \tfrac{1}{4}\left\{\sin t\int_0^t -u\,du + \int_0^t u\sin(2u-t)du + \int_0^t \cos(t-2u)du - \cos t\int_0^t du\right\}$$

$$= \tfrac{1}{4}\left\{-\sin t\tfrac{u^2}{2}\Big|_0^t + \left[-\tfrac{1}{2}u\cos(2u-t)\Big|_0^t + \tfrac{1}{2}\int_0^t \cos(2u-t)du\right] - \tfrac{1}{2}\sin(t-2u)\Big|_0^t - \cos tu\Big|_0^t\right\}$$

Note: The second integral was integrated by parts, where $U = u$, $dV = \sin(2u - t)du$, so: $dU = du$, $V = -{}^1/_2\cos(2u - t)$.

$$= \tfrac{1}{4}\left\{\sin t\left(-\tfrac{1}{2}t^2\right) - \tfrac{1}{2}t\cos t + \tfrac{1}{4}\sin(2u-t)\Big|_0^t + \tfrac{1}{2}\sin t + \tfrac{1}{2}\sin t - t\cos t\right\}$$

$$= \tfrac{1}{4}\left\{-\tfrac{1}{2}t^2\sin t - \tfrac{1}{2}t\cos t + \tfrac{1}{4}\sin t + \tfrac{1}{4}\sin t + \sin t - t\cos t\right\}$$

$$= \tfrac{1}{4}\left\{-\tfrac{1}{2}t^2\sin t - \tfrac{3}{2}t\cos t + \tfrac{3}{2}\sin t\right\} = \tfrac{1}{8}\left(-t^2\sin t - 3t\cos t + 3\sin t\right)$$

Hence: $L^{-1}\left\{\frac{1}{(s^2+1)^3}\right\} = \frac{1}{8}\left[\left(3-t^2\right)\sin t - 3t\cos t\right]$

b.) To obtain the inverse LT of the fourth term of the Laplace transformed ODE, i.e., the $\frac{4s-2}{(s^2+1)^2}$ term, apply Lemma2 Handout 5b:

If: $F(s) = \dfrac{N(s)}{\left[(s-a)^2 + b^2\right]^2}$,

then: $L^{-1}\{F(s)\} = \frac{e^{at}}{2b^3}\left[(q - bp^* - bpt)\cos bt + (p + bq^* + bqt)\sin bt\right]$

where: $k = a + ib \Rightarrow R(s) = N(s), R(k) = p + iq, R'(k) = p^* + iq^*$

Here:

$$F(s) = \frac{4s-2}{\left[(s-0)^2 + 1^2\right]^2} \Rightarrow a = 0, b = 1, k = a + ib = i$$

$$R(s) = N(s) = 4s - 2 \Rightarrow R(k) = 4i - 2 = -2 + 4i \Rightarrow p = -2, q = 4$$

$$R'(s) = 4 \Rightarrow R'(k) = 4 \Rightarrow 4 + 0i \Rightarrow p^* = 4, q^* = 0$$

So: $L^{-1}\{F(s)\} = \frac{e^{0t}}{2}\left[(4 - 4 + 2t)\cos t + (-2 + 0 + 4t)\sin t\right] = \frac{1}{2}\left[2t\cos t + (-2 + 4t)\sin t\right]$

Hence: $L^{-1}\left\{\frac{4s-2}{(s^2+1)^2}\right\} = t\cos t + (-1 + 2t)\sin t$

c.) So tallying up:

$$x(t) = L^{-1}\{Y(s)\} = L^{-1}\left\{\frac{1}{(s^2+1)^2}\right\} + L^{-1}\left\{\frac{s}{s^2+1}\right\} - 2L^{-1}\left\{\frac{1}{s^2+1}\right\} + L^{-1}\left\{\frac{4s-2}{(s^2+1)^2}\right\}$$
$$= \tfrac{1}{8}\left(-t^2 \sin t - 3t\cos t + 3\sin t\right) + \cos t - 2\sin t + \left(t\cos t + (-1+2t)\sin t\right)$$
$$= \left(\tfrac{3}{8} - \tfrac{1}{8}t^2 - 2 - 1 + 2t\right)\sin t + \left(-\tfrac{3}{8}t + 1 + t\right)\cos t$$
$$= \left(1 + \tfrac{5}{8}t\right)\cos t - \left(\tfrac{21}{8} - 2t + \tfrac{1}{8}t^2\right)\sin t$$