MA360 **Handout 5b (week of Feb 6)**
Sp2006 Kallfelz
Note: New (absolute) due date for Assignment I (See posted assignment sheet for details)
Assignment: page 68 Sheng (Exercise 6) Problems 1,2,3,5,6,9,11,14,18,20

Partial Fractions

The core idea behind this technique is algebraic: We know how to combine sums/differences of rational expressions into one rational expression. It's the procedure of obtaining a common denominator. The question is how to *reverse* the procedure, i.e. express a rational expression in terms of the sum of simpler rational expressions: this is the technique of *partial fractions:*

← *Getting a Common Denominator*

(X.1) $$\frac{f(x)}{(x-x_1)\ldots(x-x_k)g(x)} = \frac{A_1}{(x-x_1)} + \ldots + \frac{A_k}{} + \frac{K(x)}{(x-x_0)\ \ g(x)} + \ldots$$

→ *Method of Partial Fractions*

The method is straightforward: As suggested above, we see constants $A_1 \ldots A_k$ and functions K(x) ... such that above fractions can be de-coupled into the sum of simpler fractions listed on right. The technique is *purely algebraic*: Multiply (X.1) by the denominator term. Because x is a *variable*, we can se x equal to the previous denominator factor, isolating the various constants. There are various rules of thumb to this process (See table).

- ***Note:*** In the case of non-repeating linear irreducible factors, the ***Heaviside Cover-Method*** provides a convenient short-cut: Simply set: $x = x_k$, "cover" the term on LHS, them the solution for A_k is automatically computed.

If (X.1) is of the form: *Then use:*

I.

$$\frac{f(x)}{(x-x_1)\dots(x-x_k)} = \frac{A_1}{(x-x_1)} + \dots + \frac{A_k}{(x-x_0)}.$$

(non repeating irreducible linear factors)

II.

$$\frac{f(x)}{(x-x_1)^k} = \frac{A_1}{(x-x_1)} + \dots + \frac{A_k}{(x-x_1)^k}.$$

(repeating irreducible linear factors)

III.

$$\frac{f(x)}{(\alpha_1 x^2 + \beta_1 x + \gamma_1)\dots(\alpha_k x^2 + \beta_k x + \gamma_k)} = \frac{A_1 x + B_1}{(\alpha_1 x^2 + \beta_1 x + \gamma_1)} + \dots + \frac{A_k x + B_k}{(\alpha_k x^2 + \beta_k x + \gamma_k)}$$

(non repeating irreducible quadratic factors)

IV.

$$\frac{f(x)}{(\alpha_1 x^2 + \beta_1 x + \gamma_1)^k} = \frac{A_1 x + B_1}{(\alpha_1 x^2 + \beta_1 x + \gamma_1)} + \dots + \frac{A_k x + B_k}{(\alpha_1 x^2 + \beta_1 x + \gamma_1)^k}$$

(repeating irreducible quadratic factors)

- Example (Problem 4, p 68 Sheng)

$F(s) = \frac{(3s+7)e^{-2s}}{(s^2-2s-3)} = L\{u(t-2)f(t-2)\}$ (according to Thm8) where:

$$F(s) = \frac{(3s+7)}{(s^2-2s-3)} = L\{f(t)\} = \frac{(3s+7)}{(s-3)(s+1)} = \frac{A}{(s-3)} + \frac{B}{(s+1)}$$

Using Heaviside Cover Method:

$$s_1 = 3 \Rightarrow A = \frac{(3 \cdot 3+7)}{(3+1)} = 4$$
$$s_2 = -1 \Rightarrow B = \frac{(-3+7)}{(-1-3)} = -1$$

Hence: $F(s) = \frac{(3s+7)}{(s^2-2s-3)} = \frac{4}{(s-3)} - \frac{1}{(s+1)} = 4L\{e^{3t}\} - L\{e^{-t}\} = L\{(4e^{3t} - e^{-t})\}$

So:

$$F(s)=\frac{(3s+7)e^{-2s}}{(s^2-2s-3)}=L\{u(t-2)f(t-2)\}=L\{u(t-2)(4e^{3(t-2)}-e^{-(t-2)})\}$$
$$\Rightarrow f(t)=4e^{-6}e^{3t}-e^2e^{-t},t>2$$

- Example (#7, Sheng, p 68)

$$F(s)=\frac{s+5}{(s+1)(s^2+1)}$$

Method 1: (cumbersome, without using partial fractions[1])

$$F(s)=\frac{s+5}{(s+1)(s^2+1)}=\frac{(s+1)+4}{(s+1)(s^2+1)}=\frac{(s+1)}{(s+1)(s^2+1)}+\frac{4}{(s+1)(s^2+1)}=\frac{1}{(s^2+1)}+4\frac{1}{(s+1)(s^2+1)}=L\{\sin t\}+4\frac{1}{(s+1)(s^2+1)}$$

Aside: $\frac{1}{(s+1)(s^2+1)}=\frac{1}{(s+1)((s+1)^2-2s)}=\frac{1}{(s+1)[(s+1)^2-2(s+1)+2]}\Rightarrow L\{e^{-t}f(t)\}$

$$\therefore L\{f(t)\}=\frac{1}{s[s^2-2s+2]}=\frac{1}{s}L\{g(t)\}$$

(using Thm7, Thm 4)

$$L\{g(t)\}=\frac{1}{(s^2-2s+2)}=\frac{1}{(s^2-2s+1+1)}=\frac{1}{[(s-1)^2+1]}=L\{e^th(t)\}\Rightarrow L\{h(t)\}=\frac{1}{s^2+1}=L\{\sin t\}$$
$$\therefore L\{g(t)\}=L\{e^t\sin t\}\Rightarrow L\{f(t)\}=\frac{1}{s}L\{e^t\sin t\}=L\left\{\int_0^t e^{\omega}\sin\omega d\omega\right\}$$
$$\Rightarrow f(t)=\int_0^t e^{\omega}\sin\omega d\omega=\frac{e^{\omega}}{1^2+1^2}(\sin\omega-\cos\omega)\Big|_0^t=\frac{1}{2}e^t(\sin t-\cos t)-\frac{1}{2}e^0(0-1)$$
$$=\frac{1}{2}\{e^t(\sin t-\cos t)+1\}\Rightarrow\frac{1}{(s+1)(s^2+1)}=L\{e^{-t}\tfrac{1}{2}[e^t(\sin t-\cos t)+1]\}$$
$$\Rightarrow L^{-1}\left\{\frac{1}{(s+1)(s^2+1)}\right\}=\frac{1}{2}[(\sin t-\cos t)+e^{-t}]$$

so: $F(s)=L\{\sin t\}+4\frac{1}{(s+1)(s^2+1)}=L\{\sin t+2\sin t-2\cos t+2e^{-t}\}=L\{3\sin t-2\cos t+2e^{-t}\}$

$$\therefore f(t)=3\sin t-2\cos t+2e^{-t}$$

Method 2: (Using partial fractions)

- **Method 2a: (Resolving the irreducible over the complex numbers)**

$$\frac{s+5}{(s+1)(s^2+1)}=\frac{s+5}{(s+1)(s-i)(s+i)}=\frac{A}{(s+1)}+\frac{B}{(s-i)}+\frac{C}{(s+i)}$$

[1] In some of these problems, especially of this garden variety, there exists more than one approach. It's valuable to see more than one method applied.

Using Heaviside: $s_1 = -1 \Rightarrow A = \frac{-1+5}{(-1-i)(-1+i)} = \frac{4}{(1+1)} = 2$

$s_2 = i \Rightarrow B = \frac{i+5}{(i+1)(i+i)} = \frac{i+5}{2i(i+1)} = \frac{i+5}{2i-2} = \frac{1}{2}\left(\frac{i+5}{i-1}\right) = \frac{1}{2}\left(\frac{i+5}{i-1} \cdot \frac{i+1}{i+1}\right) = \frac{1}{2}\left(\frac{-1+6i+5}{-1-1}\right) = -\frac{1}{4}(4+6i) = -1-\frac{3}{2}i$

$s_3 = -i \Rightarrow C = \frac{-i+5}{(-i+1)(-i-i)} = \frac{-i+5}{-2i(-i+1)} = \frac{-i+5}{2-2i} = \frac{1}{2}\left(\frac{-i+5}{1-i}\right) = \frac{1}{2}\left(\frac{-i+5}{1-i} \cdot \frac{1+i}{1+i}\right) = \frac{1}{2}\left(\frac{1+4i+5}{1+1}\right) = \frac{1}{4}(4+6i) = 1+\frac{3}{2}i$

Hence: $\frac{s+5}{(s+1)(s^2+1)} = \frac{s+5}{(s+1)(s-i)(s+i)} = \frac{A}{(s+1)} + \frac{B}{(s-i)} + \frac{C}{(s+i)} = \frac{2}{(s+1)} - \frac{1+\frac{3}{2}i}{(s-i)} + \frac{1+\frac{3}{2}i}{(s+i)}$

$F(s) = \frac{2}{(s+1)} - \frac{1+\frac{3}{2}i}{(s-i)} + \frac{1+\frac{3}{2}i}{(s+i)} = 2L\{e^{-t}\} - (1+\frac{3}{2}i)L\{e^{it}\} + (1+\frac{3}{2}i)L\{e^{-it}\}$

So: $= L\{2e^{-t} - (1+\frac{3}{2}i)e^{it} + (1+\frac{3}{2}i)e^{-it}\} \Rightarrow f(t) = 2e^{-t} - (e^{it} - e^{-it}) - \frac{3}{2}i(e^{it} - e^{-it})$

$= 2e^{-t} - 2i\sin t - \frac{3}{2}i[2i\sin t] = 2e^{-t} - 2i\sin t + 3\sin t$

Note: $i\sin t = \cos t$, since $i = e^{i\frac{\pi}{2}}$, which rotates any complex number $z = x + iy = re^{i\phi}$ (where: $r = \sqrt{x^2 + y^2}, \phi = \arctan\left(\frac{y}{s}\right)$) in the counterclockwise direction by $\frac{\pi}{2}$. Hence, when $z = \sin t = y \Rightarrow e^{i\frac{\pi}{2}} z = \sin\left(t + \frac{\pi}{2}\right) = \cos t$

So: $f(t) = 2e^{-t} - 2\cos t + 3\sin t$

- **Method 2b: Focusing on the real numbers only (leaving second term in denominator in irreducible form)**

$$\frac{s+5}{(s+1)(s^2+1)} = \frac{s+5}{(s+1)(s^2+1)} = \frac{A}{(s+1)} + \frac{B_1 s + B_2}{(s^2+1)} \Rightarrow s+5 = A(s^2+1) + (B_1 s + B_2)(s+1)$$

$$\Rightarrow s + 5 = (A + B_1)s^2 + (B_1 + B_2)s + (A + B_2) \Rightarrow A = -B_1 \Rightarrow \begin{matrix} B_1 + B_2 = 1 \\ -B_1 + B_2 = 5 \end{matrix}$$

$$\Rightarrow B_2 = 3, B_1 = -2$$

Hence:

$$\frac{s+5}{(s+1)(s^2+1)} = \frac{s+5}{(s+1)(s^2+1)} = \frac{2}{(s+1)} + \frac{-2s+3}{(s^2+1)} = \frac{2}{(s+1)} - 2\frac{s}{(s^2+1)} + 3\frac{1}{(s^2+1)} = 2L\{e^{-t}\} - 2L\{\cos t\} + 3L\{\sin t\}$$

$$= L\{2e^{-t} - 2\cos t + 3\sin t\} \therefore f(t) = 2e^{-t} - 2\cos t + 3\sin t$$

- Example (Sheng, problem 12)

$$F(s)=\frac{2s^2-3s}{(s-2)(s-1)^2}=\frac{s(2s-3)}{(s-2)(s-1)^2}=\frac{A}{(s-2)}+\frac{B_1}{(s-1)}+\frac{B_2}{(s-1)^2}\Rightarrow s(2s-3)=A(s-1)^2+B_1(s-2)(s-1)+B_2(s-2)$$

$$s_1=1\Rightarrow B_2=1$$
$$s_2=2\Rightarrow A=2$$
$$\therefore (2s^2-3s)=2(s-1)^2+B_1(s-2)(s-1)+(s-2)$$
$$\Rightarrow \tfrac{d}{ds}(2s^2-3s)=4(s-1)+B_1[(s-2)+(s-1)]+1$$
$$4s-3=4s-4+(2s-3)B_1+1$$
$$\Rightarrow 4s-3=4s-3+B_1(2s-3)\Rightarrow B_1=0$$

$$F(s)=\frac{2}{(s-2)}+\frac{1}{(s-1)^2}=2L\{e^{2t}\}+G(s-1)$$

So: $\Rightarrow G(s)=\frac{1}{s^2}=L\{t\}\therefore G(s-1)=L\{e^t t\}$

$$\therefore F(s)=2L\{e^{2t}\}+L\{e^t t\}=L\{2e^{2t}+te^t\}\Rightarrow f(t)=e^t(2e^t+t)$$

Lemma1: (From Example 3-3, p. 63, Sheng): $L^{-1}\left\{\frac{s+a}{(s+b)^2+\omega^2}\right\}=\frac{k}{\omega}e^{-bt}\sin(\omega t+\phi)$

(where: $k=\sqrt{(a-b)^2+\omega^2},\phi=\arctan\left(\frac{\omega}{a-b}\right)$)

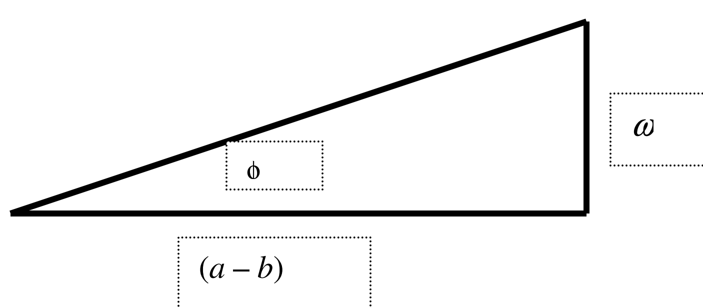
Proof: $\frac{s+a}{(s+b)^2+\omega^2}=\frac{s+b+(a-b)}{(s+b)^2+\omega^2}=\frac{(s+b)}{(s+b)^2+\omega^2}+\frac{(a-b)}{(s+b)^2+\omega^2}=F(s+b)+(a-b)G(s+b)$

Where: $F(s)=\frac{s}{s^2+\omega^2}=L\{\cos\omega t\},G(s)=\frac{1}{s^2+\omega^2}=\frac{1}{\omega}\cdot\frac{\omega}{s^2+\omega^2}=\frac{1}{\omega}L\{\sin\omega t\}$

Hence:

$$\frac{s+a}{(s+b)^2+\omega^2}=F(s+b)+(a-b)G(s+b)=L\{e^{-bt}\cos\omega t\}+\frac{(a-b)}{\omega}L\{e^{-bt}\sin\omega t\}$$
$$=L\left\{e^{-bt}\left[\cos\omega t+\frac{(a-b)}{\omega}\sin\omega t\right]\right\}\Rightarrow f(t)=e^{-bt}\left[\cos\omega t+\frac{(a-b)}{\omega}\sin\omega t\right]$$
$$=\frac{e^{-bt}}{\omega}[\omega\cos\omega t+(a-b)\sin\omega t]$$

Consider the following triangle, as suggested by the coefficients in the above terms:



Then according to the Pythagorean Theorem, its hypotenuse is: $k = \sqrt{(a-b)^2 + \omega^2}$

And: $\phi = \tan^{-1}\left(\frac{\omega}{a-b}\right), \cos\phi = \frac{a-b}{k}, \sin\phi = \frac{\omega}{k}$

So:

$$\frac{s+a}{(s+b)^2+\omega^2} = \frac{e^{-bt}}{\omega}\left[\omega\cos\omega t + (a-b)\sin\omega t\right] = \frac{ke^{-bt}}{\omega}\left[\frac{\omega}{k}\cos\omega t + \frac{(a-b)}{k}\sin\omega t\right]$$
$$= \frac{ke^{-bt}}{\omega}(\sin\phi\cos\omega t + \cos\phi\sin\omega t) = \frac{ke^{-bt}}{\omega}\sin(\omega t + \phi)$$

Lemma2: (From Example 3-4, p. 65, Sheng):

If: $F(s) = \frac{N(s)}{\left[(s-a)^2 + b^2\right]^2}$,

then: $L^{-1}\{F(s)\} = \frac{e^{at}}{2b^3}\left[(q - bp^* - bpt)\cos bt + (p + bq^* + bqt)\sin bt\right]$

where: $k = a + ib \Rightarrow R(s) = N(s), R(k) = p + iq, R'(k) = p^* + iq^*$

(**Note:** p^*, q^* are not the complex conjugates of p and q! The superscript just indicates that they're not necessarily equal to p , q.)

Proof (Sketch)[2]:

$F(s) = \frac{N(s)}{\left[(s-a)^2 + b^2\right]^2} = (r(s-a))^{-4} N(s)$, where: $r(s-a) = \sqrt{(s-a)^2 + b}$, such that for $z = x + iy = r(s)e^{i\varphi} \Rightarrow x(s) = \frac{s}{r(s)} = \cos\varphi, y = \frac{b}{r(s)} = \sin\varphi, \varphi(s) = \tan^{-1}\left(\frac{b}{s}\right)$

$$\frac{N(s)}{\left[(s-a)^2 + b^2\right]^2} = \frac{A_1 s + A_2}{\left[(s-a)^2 + b^2\right]} + \frac{B_1 s + B_2}{\left[(s-a)^2 + b^2\right]^2}$$
$$\Rightarrow N(s) = (A_1 s + A_2)\left((s-a)^2 + b^2\right) + B_1 s + B_2$$

(**Note**: the coefficients A_1, A_2, B_1, B_2 are all real-valued)

Note that: $N(k) = N(a + ib) = [A_1 k + A_2]\left[(-ib)^2 + b^2\right] + B_1(a + ib) + B_2$

Hence:
$$N(k) = N(a + ib) = [A_1 k + A_2]\left[-b^2 + b^2\right] + B_1(a + ib) + B_2$$
$$\Rightarrow N(k) = B_1 a + B_2 + ibB_1$$

[2] This proof comprises lots of tedious algebra, I set up the strategy here and omit the details near the end.

Define: $N(k)=R(k)=p+iq$, then: $p+iq=(aB_1+B_2)+ibB_1 \Rightarrow B_1=\frac{q}{b}, B_2=\frac{pb-aq}{b}$

Furthermore: $N'(s)=A_1\left((s-a)^2+b\right)+2(A_1 s+A_2)(s-a)+\frac{q}{b}$

(substituting in the value for $B_1=\frac{q}{b}$)

Define: $N'(k)=N'(a+ib)=p^*+iq^*$

Hence:

$$N'(k)=A_1\left((ib)^2+b\right)+2(A_1(a+ib)+A_2)(a+ib-a)+\frac{q}{b}=2i(aA_1+A_2)b-2b^2A_1+\frac{q}{b}$$
$$\Rightarrow p^*+iq^*=\left(\frac{q}{b}-2b^2A_1\right)+2i(aA_1+A_2)b$$
$$\Rightarrow A_1=\frac{1}{2b^2}\left(\frac{q}{b}-p^*\right)$$

(Equating real terms on both sides to get A_1)

Moreover, when equating imaginary terms on both sides:

$$q^*=2baA_1+2A_2=\frac{a}{b}\left(\frac{q}{b}-p^*\right)+2bA_2 \Rightarrow A_2=\frac{1}{2b}\left[\left(q^*+\frac{a}{b}p^*\right)-\frac{aq}{b^2}\right]$$

$$F(s)=\frac{N(s)}{\left[(s-a)^2+b^2\right]^2}=\frac{A_1 s+A_2}{\left[(s-a)^2+b^2\right]}+\frac{B_1 s+B_2}{\left[(s-a)^2+b^2\right]^2}$$
$$=A_1\frac{s}{\left[(s-a)^2+b^2\right]}+A_2\frac{1}{\left[(s-a)^2+b^2\right]}+B_1\frac{s}{\left[(s-a)^2+b^2\right]^2}+B_2\frac{1}{\left[(s-a)^2+b^2\right]^2}$$

- Use **Lemma1** for the first term: $\frac{s}{\left[(s-a)^2+b^2\right]}$ (setting the constants in **Lemma1** equal to the constants here in the following fashion: $a = 0$, $b = -a$, $\omega = b$

  Hence:

$$L^{-1}\left\{\frac{s}{(s-a)^2+b^2}\right\}=\frac{\sqrt{a^2+b^2}}{b}e^{at}\left(\sin bt+\arctan\left(-\frac{b}{a}\right)\right)=\frac{\sqrt{a^2+b^2}}{b}e^{at}\left(\sin bt\cos\phi+\cos bt\sin\phi\right)$$
$$=\frac{\sqrt{a^2+b^2}}{b}e^{at}\left(\sin bt\frac{-a}{\sqrt{a^2+b^2}}+\cos bt\frac{b}{\sqrt{a^2+b^2}}\right)=\frac{e^{at}}{b}\left(-a\sin bt+b\sin bt\right)$$

- The second term of course is (using Thm7):

$$L^{-1}\left\{\frac{1}{(s-a)^2+b^2}\right\}=e^{at}L^{-1}\left\{\frac{1}{s^2+b^2}\right\}=\frac{e^{at}}{b}L^{-1}\left(\frac{b}{s^2+b^2}\right)=\frac{e^{at}}{b}\sin bt$$

- For the third term:

$$\frac{s}{\left[(s-a)^2+b^2\right]^2}=\frac{s-a}{\left[(s-a)^2+b^2\right]^2}+\frac{a}{\left[(s-a)^2+b^2\right]^2}$$

For the term: $\frac{s-a}{\left[(s-a)^2+b^2\right]^2} = H(s-a) \Rightarrow H(s) = \frac{s}{\left(s^2+b^2\right)^2}$, where $L^{-1}\{H(s)\}$ can be obtained either through THM12 or by Partial Fraction

- For the fourth and the other part of the third term: $\frac{1}{\left[(s-a)^2+b^2\right]^2}, \frac{a}{\left[(s-a)^2+b^2\right]^2}$

Note: $\frac{a}{\left[(s-a)^2+b^2\right]^2} = G(s-a) \Rightarrow G(s) = \frac{a}{\left(s^2+b^2\right)^2}$, which can be resolved through partial fractions.

- <u>Example (Problem 8, p 68 Sheng)</u>

$$F(s) = \frac{2s}{\left(s^2-10s-50\right)}$$

the denominator term is reducible, since $b^2 - 4ac = 100 + 200 = 300 > 0$

Method 1: Factor denominator and use partial fractions

$$s_{1,2} = \frac{-(-10) \pm \sqrt{100+200}}{2} = 5 \pm \frac{\sqrt{300}}{2} = 5 \pm 5\sqrt{3}$$

$$\frac{2s}{\left(s^2-10s-50\right)} = \frac{2s}{(s-s_1)(s-s_2)} = \frac{A}{(s-s_1)} + \frac{B}{(s-s_2)}$$

Using Heaviside Cover method: $s = s_1 \Rightarrow A = \frac{2s_1}{(s_1-s_2)} = \frac{10\left(1+\sqrt{3}\right)}{10\sqrt{3}} = \frac{1+\sqrt{3}}{\sqrt{3}} = \frac{1}{3}\left(3+\sqrt{3}\right)$

$$s = s_2 \Rightarrow B = \frac{2s_2}{(s_2-s_1)} = \frac{10\left(1-\sqrt{3}\right)}{-10\sqrt{3}} = \frac{-1+\sqrt{3}}{\sqrt{3}} = \frac{1}{3}\left(3-\sqrt{3}\right)$$

$$\therefore F(s) = \frac{2s}{\left(s^2-10s-50\right)} = \frac{1}{3}\left\{\left(3+\sqrt{3}\right)\frac{1}{s-s_1} + \left(3-\sqrt{3}\right)\frac{1}{s-s_2}\right\}$$
$$\Rightarrow f(t) = \frac{1}{3}\left(3+\sqrt{3}\right)e^{5\left(1+\sqrt{3}\right)t} + \frac{1}{3}\left(3-\sqrt{3}\right)e^{5\left(1-\sqrt{3}\right)t}$$
$$= \frac{e^{5t}}{3}\left\{3\left(e^{5\sqrt{3}t} + e^{-5\sqrt{3}t}\right) + \sqrt{3}\left(e^{5\sqrt{3}t} - e^{5\sqrt{3}t}\right)\right\}$$

To confirm identity with Sheng's answer, observe:

$$= \frac{e^{5t}}{3}\left\{3\left(e^{5\sqrt{3}t} + e^{-5\sqrt{3}t}\right) + \sqrt{3}\left(e^{5\sqrt{3}t} - e^{5\sqrt{3}t}\right)\right\}$$
$$= \frac{e^{5t}}{3}\left\{6\cosh 5\sqrt{3}t + 2\sqrt{3}\sinh 5\sqrt{3}t\right\} = 2e^{5t}\cosh 5\sqrt{3}t + \frac{2}{3}e^{5t}\sqrt{3}\sinh 5\sqrt{3}t$$
$$= 2e^{5t}\cosh\sqrt{25\cdot 3}t + \frac{2}{\sqrt{3}}e^{5t}\sinh\sqrt{25\cdot 3}t = 2e^{5t}\cosh\sqrt{75}t + \frac{2\cdot 5}{\sqrt{3}\cdot 5}e^{5t}\sinh\sqrt{75}t$$
$$= 2e^{5t}\cosh\sqrt{75}t + \frac{10}{\sqrt{75}}e^{5t}\sinh\sqrt{75}t$$

Method 2: Use Formula from Lemma1:

(Completing the square in the denominator term)

$$F(s)=\frac{2s}{(s^2-10s-50)}=2\left[\frac{s+0}{s^2-10s+25-25-50}\right]=2\left[\frac{s}{(s-5)^2-75}\right]=2\left[\frac{s-0}{(s-5)^2+\left(i\sqrt{75}\right)^2}\right]$$

Hence: $\omega=\sqrt{75}i, a=0, b=5$

So: $f(t)=\frac{k}{\omega}e^{5t}\sin(\omega t+\phi)$

Where: $k=\sqrt{(0-5)^2+\left(i\sqrt{75}\right)^2}=\sqrt{25-75}=5i\sqrt{2}$

Hence:
$$f(t)=\frac{k}{\omega}e^{5t}\sin(\omega t+\phi)=\frac{k}{\omega}e^{5t}\left(\sin i\sqrt{75}\cos\phi+\cos i\sqrt{75}t\sin\phi\right)$$
$$=\frac{i\sqrt{50}}{i\sqrt{75}}e^{5t}\left(\sin i\sqrt{75}t\cdot\frac{(0-5)}{i\sqrt{50}}+\cos i\sqrt{75}t\cdot\frac{i\sqrt{75}}{i\sqrt{50}}\right)$$

(To obtain the expressions for sinϕ, cosϕ, recall the triangle in the proof of Lemma 1, sketched above in page 5)

$$f(t)=\frac{i\sqrt{50}}{i\sqrt{75}}e^{5t}\left(\sin i\sqrt{75}t\cdot\frac{(0-5)}{i\sqrt{50}}+\cos i\sqrt{75}t\cdot\frac{i\sqrt{75}}{i\sqrt{50}}\right)$$
$$=\frac{e^{5t}}{i\sqrt{75}}\left\{\frac{5}{2i}\left(e^{i\left(i\sqrt{75}\right)t}-e^{-i\left(i\sqrt{75}\right)t}\right)+\frac{i\sqrt{75}}{2}\left(e^{i\left(i\sqrt{75}\right)t}+e^{-i\left(i\sqrt{75}\right)t}\right)\right\}$$
$$=\frac{e^{5t}}{\sqrt{75}}\left\{-\frac{5}{2}\left(e^{-\sqrt{75}t}-e^{\sqrt{75}t}\right)+\frac{\sqrt{75}}{2}\left(e^{-\sqrt{75}t}+e^{\sqrt{75}t}\right)\right\}$$
$$=\frac{e^{5t}}{\sqrt{75}}\left\{\frac{5}{2}\sinh\sqrt{75}t+\frac{\sqrt{75}}{2}\cosh\sqrt{75}t\right\}$$

then multiply final answer by the coefficient 2

Method3 (Using techniques from previous chapter)

$$F(s)=\frac{2s}{(s^2-10s-50)}=2\left[\frac{s}{(s-5)^2-75}\right]=2\left[\frac{s-5+5}{(s-5)^2-\left(\sqrt{75}\right)^2}\right]=2\left\{\frac{s-5}{(s-5)^2-\sqrt{75^2}}+\frac{5}{(s-5)-\sqrt{75^2}}\right\}$$

Aside: $\frac{s-5}{(s-5)^2-\sqrt{75^2}}=G(s-5)=L\left\{e^{5t}g(t)\right\}\Rightarrow g(t)=L^{-1}\left\{\frac{s}{s^2-\sqrt{75^2}}\right\}=\cosh\sqrt{75}t$

$$\frac{5}{(s-5)^2-\sqrt{75^2}}=H(s-5)=L\left\{e^{5t}h(t)\right\}\Rightarrow h(t)=L^{-1}\left\{\frac{5}{s^2-\sqrt{75^2}}\right\}=\frac{5}{\sqrt{75}}\sinh\sqrt{75}t$$

Hence: $F(s)=2\left\{\frac{s-5}{(s-5)^2-\sqrt{75^2}}+\frac{5}{(s-5)-\sqrt{75^2}}\right\}=L\left\{e^{5t}2\cosh\sqrt{75}t+\frac{10}{\sqrt{75}}\sinh\sqrt{75}t\right\}$

- <u>Example (Problem 20, Sheng)</u>

$$F(s) = \frac{s^3+3s^2-s-3}{\left(s^2+2s+5\right)^2} = \frac{N(s)}{\left[(s+1)^2+2^2\right]^2}$$

Using **Lemma2:** a = -1, b = 2, $k = a + ib$ = -1+2i

$$\therefore N(s) = s^3 + 3s^2 - (s+3) = s^2(s+3) - (s+3) = \left(s^2-1\right)(s+3) = (s-1)(s+1)(s+3)$$
$$N'(s) = 3s^2 + 6s - 1$$
$$\Rightarrow N(k) = p + iq = (-1+2i-1)(-1+2i+1)(-1+2i+3) = (-2+2i)(2i)(2+2i)$$
$$= 8(-1+i)(i)(1+i) = 8i(i-1)(1+i) = -16i$$

so: p = 0, q = -16

$$N'(s) = 3s^2 + 6s - 1$$
$$\Rightarrow N'(k) = p^* + iq^* = 3(-1+2i)^2 + 6(-1+2i) - 1 = 3(1-4i-4) - 6 + 12i - 1 = 3 - 12i - 12 - 6 + 12i - 1 = -16$$

so: p^* = -16, q^* = 0

hence:

$$L^{-1}\{F(s)\} = \frac{e^{at}}{2b^3}\left[(q - bp^* - bpt)\cos bt + (p + bq^* + bqt)\sin bt\right]$$
$$= \frac{e^{-t}}{16}\left[(-16+32-0)\cos 2t + (0+0-32t)\sin 2t\right]$$
$$= \frac{e^{-t}}{16}\left[16\cos 2t - 32t\sin 2t\right] = e^{-t}\left(\cos 2t - 2t\sin 2t\right)$$