MA360 **Handout 5a (week of Jan 6)**
Sp2006 Kallfelz
Note: New (absolute) due date for Assignment I (See posted assignment sheet for details)
Assignment: page 68 Sheng (Exercise 6) Problems 1,2,3,5,6,9,11,14,18,20

- Example (3(a), p54 Sheng, modified)

  Find $f(t), f'(t)$ and the associated LTs of $f(t), f'(t)$:

  a.) From the graph:

$$f(t)=\begin{cases} 2t+2 & -1\le t<0 \\ -2t+2 & 0\le t<1 \end{cases} = 2(t+1)(u(t+1)-u(t))+2(1-t)(u(t)-u(t-1))$$
$$=2\{(t+1)u(t+1)-tu(t)-u(t)+u(t)-tu(t)-(t-1)u(t-1)\}$$
$$=2\{(t+1)u(t+1)-2tu(t)-(t-1)u(t-1)\}$$

b.)

$$L\{f(t)\}=2L\{(t+1)u(t+1)\}-4L\{tu(t)\}-2L\{(t-1)u(t-1)\}$$
$$=2e^{s}\cdot\tfrac{1}{s^2}-4e^{-0s}\cdot\tfrac{1}{s^2}-2e^{-s}\cdot\tfrac{1}{s^2}=\tfrac{2}{s^2}\left(e^{s}-2-e^{-s}\right)$$

(Note: It's obviously much simpler to use THM8 for the above three terms, where, in each case, the shifting constant is $a = 1$, $a = 0$, $a = -1$, respectively, and in all three above cases, $f(t) = t$.)

c.)

$$f'(t)=\tfrac{d}{dt}2\{(t+1)u(t+1)-2tu(t)-(t-1)u(t-1)\}$$
$$=2\{u(t+1)+(t+1)\delta(t+1)-2u(t)-2t\delta(t)-u(t-1)-(t-1)\delta(t-1)\}$$
$$=2\{u(t+1)-2u(t)-u(t-1)\}+2\{t\delta(t+1)+\delta(t+1)-2t\delta(t)-t\delta(t-1)+\delta(t-1)\}$$
$$=2\{[u(t+1)-2u(t)-u(t-1)]+[t\delta(t+1)+\delta(t+1)-t\delta(t-1)+\delta(t-1)]\}$$

(recall that $t\delta(t) = 0$ for all t)

d.) According to THM2: $L\{f'(t)\}=sL\{f(t)\}-f(0)$

Here: $f(0)=2\{(0+1)u(1)-2\cdot 0-(0-1)u(0-1)\}=2$

(**Note:** we can see this directly from inspecting the graph of this function. But analytically, note that according to the definition of the step function, the first term survives)

So: $L\{f'(t)\}=s\cdot\tfrac{2}{s^2}\left(e^{s}-2-e^{-s}\right)-2=\tfrac{2}{s}\left[e^{s}-2-s-e^{-s}\right]$

Pre-Test Exercises (Answers)

1. Use Euler's Thm. to derive: $L\{\cos \omega t\}, L\{\sin \omega t\}$

$$L\{e^{i\omega t}\} = \tfrac{1}{s-i\omega t} = \tfrac{1}{s-i\omega t}\cdot\tfrac{s+i\omega}{s+i\omega} = \tfrac{s+i\omega}{s^2+\omega^2} = \tfrac{s}{s^2+\omega^2} + i\,\tfrac{\omega}{s^2+\omega^2}$$
$$L\{e^{i\omega t}\} = L\{\cos \omega t + i\sin \omega t\} = L\{\cos \omega t\} + iL\{\sin \omega t\}$$
$$\therefore L\{\cos \omega t\} = \tfrac{s}{s^2+\omega^2},\ L\{\sin \omega t\} = \tfrac{\omega}{s^2+\omega^2}$$

2. a.) Find $L\{f(t)\}, f(t) = \left\{\begin{matrix} t-1 & 1 \le t \le 2 \\ 0 & otherwise \end{matrix}\right\}$ from first principles

$$L\{f\} = \int_0^\infty f(t)e^{-ts}\,dt = \int_1^2 (t-1)e^{-ts}\,dt + \int_2^\infty 0\cdot e^{-ts}\,dt = \int_1^2 te^{-st}\,dt - \int_1^2 e^{-st}\,dt$$
$$= -\tfrac{1}{s}te^{-st}\Big|_1^2 + \tfrac{1}{s}\int_1^2 e^{-st}\,dt - \int_1^2 e^{-st}\,dt = -\tfrac{1}{s}\left(2e^{-2s} - e^{-s}\right) + \left(\tfrac{1}{s} - 1\right)\int_1^2 e^{-st}\,dt$$
$$= -\tfrac{1}{s}\left(2e^{-2s} - e^{-s}\right) - \left(\tfrac{1}{s} - 1\right)\tfrac{1}{s}e^{-st}\Big|_1^2 = \tfrac{e^{-s}}{s}\left(1 - 2e^{-s}\right) + \left(\tfrac{1}{s} - \tfrac{1}{s^2}\right)\left(e^{-2s} - e^{-s}\right)$$
$$= \tfrac{e^{-s}}{s}\left(1 - 2e^{-s}\right) + \tfrac{e^{-s}}{s}\left(1 - \tfrac{1}{s}\right)\left(e^{-s} - 1\right) = \tfrac{e^{-s}}{s}\left[1 - 2e^{-s} + e^{-s} - \tfrac{1}{s}e^{-s} + \tfrac{1}{s} - 1\right]$$
$$= \tfrac{e^{-s}}{s}\left[-e^{-s} + \tfrac{1}{s}\left(1 - e^{-s}\right)\right] = \tfrac{e^{-s}}{s^2}\left[-e^{-s}(1+s) + 1\right]$$

(The first integral above was evaluated by parts, where: $U = t$, $dV = e^{-st}$)

b.) Verify your answer in a.) using step function method

$$f(t) = \left\{\begin{matrix} t-1 & 1 \le t \le 2 \\ 0 & otherwise \end{matrix}\right\} = (t-1)\left(u(t-1) - u(t-2)\right) = (t-1)u(t-1) - tu(t-2) + u(t-2)$$

$$\therefore L\{f(t)\} = L\{(t-1)u(t-1)\} - L\{tu(t-2)\} + L\{u(t-2)\}$$

- According to THM8: $L\{(t-1)u(t-1)\} = L\{u(t-1)(t-1)\} = e^{-s}L\{t\} = \frac{e^{-s}}{s^2}$
- According to THM5: $$L\{tu(t-2)\} = (-1)^1\tfrac{d}{ds}L\{u(t-2)\} = -\tfrac{d}{ds}\left\{\tfrac{e^{-2s}}{s}\right\}$$ $$= -\tfrac{-2se^{-2s} - e^{-2s}}{s^2} = \tfrac{2e^{-2s} + e^{-2s}}{s^2} = \tfrac{e^{-2s}}{s^2}(1+2s)$$

$$\therefore L\{f(t)\} = L\{(t-1)u(t-1)\} - L\{tu(t-2)\} + L\{u(t-2)\} = \tfrac{e^{-s}}{s^2} - \tfrac{e^{-2s}}{s^2}(1+2s) + \tfrac{e^{-2s}}{s}$$
$$= \tfrac{e^{-s}}{s^2}\left[1 - e^{-s}(1+2s) + se^{-s}\right] = \tfrac{e^{-s}}{s^2}\left[-e^{-s}(1+s) + 1\right]$$

3. Find: $L\left[\frac{1}{t}\left(e^{-5t}-e^{-t}\right)\right]$

According to Thm 6: $L\left\{t^{-1}f(t)\right\}=\int_{s}^{\infty}F(\omega)d\omega$ where: $L\{f(t)\} = F(s)$

$$L\left[\frac{e^{-5t}}{t}\right]-L\left[\frac{e^{-t}}{t}\right]=\int_{s}^{\infty}L\left[e^{-5t}\right]d\omega-\int_{s}^{\infty}L\left[e^{-t}\right]d\omega$$

$$=\int_{s}^{\infty}\frac{d\omega}{\omega+5}-\int_{s}^{\infty}\frac{d\omega}{\omega+1}=\lim_{d\to\infty}\left\{\ln\left|\omega+5\right|\Big|_{s}^{d}-\ln\left|\omega+1\right|\Big|_{s}^{d}\right\}$$

$$=\lim_{d\to\infty}\ln\left|\frac{\omega+5}{\omega+1}\right|\Big|_{s}^{d}=\lim_{d\to\infty}\ln\left|\frac{d+5}{d+1}\right|-\ln\left|\frac{s+5}{s+1}\right|$$

$$=\lim_{d\to\infty}\ln\left|\frac{1+\frac{5}{d}}{1+\frac{1}{d}}\right|-\ln\left|\frac{s+5}{s+1}\right|=\ln\left|\frac{1+0}{1+0}\right|+\ln\left(\left|\frac{s+5}{s+1}\right|^{-1}\right)$$

$$=\ln 1+\ln\left|\frac{s+5}{s+1}\right|=\ln\left|\frac{s+5}{s+1}\right|$$

4. Find: $L\left[t\left(e^{t}\cosh t\right)\right]$

(Note change from cosh3*t*, to illustrate the facility of two methods below:)

Method1:

$$L\left[t\left(e^{t}\cosh t\right)\right]=L\left\{te^{t}\left(\frac{e^{t}+e^{-t}}{2}\right)\right\}=\frac{1}{2}L\left\{te^{2t}\right\}+\frac{1}{2}L\{t\}=\frac{1}{2}(-1)\frac{d}{ds}L\left\{e^{2t}\right\}+\frac{1}{2}\cdot\frac{1}{s^{2}}$$

$$=-\frac{1}{2}\frac{d}{ds}\left(\frac{1}{s-2}\right)+\frac{1}{2s^{2}}=\frac{1}{2}(s-2)^{-2}+\frac{1}{2s^{2}}=\frac{1}{2}\left[\frac{1}{(s-2)^{2}}+\frac{1}{s^{2}}\right]=\frac{1}{2}\left[\frac{s^{2}+\left(s^{2}-4s+4\right)}{s^{2}(s-2)^{2}}\right]=\frac{2s^{2}-4s+4}{2s^{2}(s-2)^{2}}$$

$$=\frac{s^{2}-2s+2}{s^{2}(s-2)^{2}}$$

Method 2:

$$L\left[t\left(e^{t}\cosh t\right)\right]=(-1)\frac{d}{ds}L\left\{e^{t}\cosh t\right\}$$

- According to THM7: $L\left\{e^{t}\cosh t\right\}=F(s-1)$, where: $F(s)=L\{\cosh t\}=\frac{s}{s^{2}-1}$, $\therefore L\left\{e^{t}\cosh t\right\}=\frac{(s-1)}{(s-1)^{2}-1}=\frac{s-1}{s^{2}-2s}$

Hence:

$$L\left[t\left(e^{t}\cosh t\right)\right]=(-1)\frac{d}{ds}L\left\{e^{t}\cosh t\right\}=-\frac{d}{ds}\left[\frac{s-1}{s(s-2)}\right]=-\left[\frac{s(s-2)-(s-1)(2s-2)}{s^{2}(s-2)^{2}}\right]$$

$$=\frac{2(s-1)^{2}-s(s-2)}{s^{2}(s-2)^{2}}=\frac{2s^{2}-4s+2-s^{2}+2s}{s^{2}(s-2)^{2}}=\frac{s^{2}-2s+2}{s^{2}(s-2)^{2}}$$

5. Find: $L\left[\int_0^t e^{\omega}\cosh 3\omega d\omega\right]$

- According to Thm4: $L\left[\int_0^t e^{-\omega}\cosh 3\omega d\omega\right] = \frac{1}{s}L\left[e^{-t}\cosh 3t\right]$

- According to Thm 7: $L[e^{-t}\cosh 3t] = F(s+1)$, where:
  $F(s) = L(\cosh at) = \frac{s}{s^2-a^2} \Rightarrow L[\cosh 3t] = \frac{s}{s^2-9}$

So: $F(s+1) = \frac{s+1}{(s+1)^2-9}$

Hence: $L\left[\int_0^t e^{-6\omega}\cosh 3\omega d\omega\right] = \frac{1}{s}\frac{(s+1)}{(s+1)^2-9} = \frac{s+1}{s\left[(s+1)^2-9\right]}$

6. Find: $L^{-1}\left[\frac{1}{s}\frac{s-2}{(s+2)^2+16}\right]$

- According to THM4: $\frac{1}{s}F(s) = L\left\{\int_0^t f(\omega)d\omega\right\}$, where: $F(s) = \frac{s-2}{(s+2)+4^2} = L\{f(t)\}$

Hence: $\frac{s-2}{(s+2)+4^2} = \frac{s+2-4}{(s+2)^2+4^2} = \frac{(s+2)}{(s+2)^2+4^2} - \frac{4}{(s+2)+4^2} = F_1(s+2) - F_2(s+2)$

- Where according to THM7:

$F_1(s) = \frac{s}{s+4^2} = L\{\cos 4t\} \Rightarrow F_1(s+2) = L\{e^{-2t}\cos 4t\}$
$F_2(s) = \frac{4}{s+4^2} = L\{\sin 4t\} \Rightarrow F_1(s+2) = L\{e^{-2t}\sin 4t\}$

Hence: $F(s) = \frac{s-2}{(s+2)+4^2} = L\{f(t)\} = L\{e^{-2t}(\cos 4t - \sin 4t)\}$

So: $\frac{1}{s}F(s) = L\left\{\int_0^t e^{-2\omega}(\cos 4\omega - \sin 4\omega)d\omega\right\}$

Hence: $L^{-1}\left[\frac{1}{s}\frac{s-2}{(s+2)^2+16}\right] = \int_0^t e^{-2\omega}(\cos 4\omega - \sin 4\omega)d\omega$

Formula 30: $\int e^{au}\sin bu du = \frac{e^{au}(a\sin bu - b\cos bu)}{b^2+a^2} + C$

Formula 31: $\int e^{au}\cos bu\,du = \dfrac{e^{au}\left(a\cos bu + b\sin bu\right)}{b^2 + a^2} + C$

So:

$$\int_0^t e^{-2\omega}\sin 4\omega d\omega = \left.\frac{e^{-2\omega}\left(-2\sin 4\omega - 4\cos 4\omega\right)}{2^2+4^2}\right|_0^t = -\tfrac{2}{20}\left[e^{-2t}\left(\sin 4t + 2\cos 4t\right) - e^0\left(0+2\right)\right]$$

$$= -\tfrac{1}{10}\left[e^{-2t}\left(\sin 4t + 2\cos 4t\right) - 2\right]$$

$$\int_0^t e^{-2\omega}\cos 4\omega d\omega = \left.\frac{e^{-2\omega}\left(-2\cos 4\omega + 4\sin 4\omega\right)}{2^2+4^2}\right|_0^t = -\tfrac{2}{20}\left[e^{-2t}\left(-\cos 4t + 2\sin 4t\right) - e^0\left(-1+0\right)\right]$$

$$= \tfrac{1}{10}\left[e^{-2t}\left(-\cos 4t + 2\sin 4t\right) + 1\right]$$

So:

$$\int_0^t e^{-2\omega}\left(\cos 4\omega - \sin 4\omega\right)d\omega = \tfrac{1}{10}\left[e^{-2t}\left(-\cos 4t + 2\sin 4t\right) + 1\right] + \tfrac{1}{10}\left[e^{-2t}\left(\sin 4t + 2\cos 4t\right) - 2\right]$$

$$= \tfrac{1}{10}\left[e^{-2t}\left(\cos 4t + 3\sin 4t\right) - 1\right]$$

7. Find: $L^{-1}\left[\frac{e^{-4s}}{(s-2)^2}\right]$

- According to THM8: $\frac{e^{-4s}}{(s-2)^2} = L\{u(t-4)f(t-4)\}$, where: $\frac{1}{(s-2)^2} = L\{f(t)\}$
- According to THM7: $\frac{1}{(s-2)^2} = F(s-2) = L\{e^{2t}g(t)\}$, where: $\frac{1}{s^2} = L\{g(t)\}$

Hence: $\frac{1}{s^2} = L\{g(t)\} = L\{t\} \Rightarrow \frac{1}{(s-2)^2} = L\{te^{2t}\} \Rightarrow \frac{e^{-4s}}{(s-2)^2} = L\{u(t-4)[(t-4)e^{2(t-4)}]\}$

So: $L^{-1}\left[\frac{e^{-4s}}{(s-2)^2}\right] = u(t-4)(t-4)e^{2(t-4)} = e^{-8}(t-4)e^{2t}, t \geq 4$

8. Find : L For |cos*t*|, $p = \pi$.

$$L[|\cos t|] = \frac{1}{1-e^{-\pi s}}\int_0^\pi e^{-st}\left(\frac{e^{it}+e^{-it}}{2}\right)dt = \frac{1}{2\left(1-e^{-\pi s}\right)}\left\{\int_0^\pi e^{(i-s)t}dt + \int_0^\pi e^{-(i+s)t}dt\right\}$$

$$= \frac{1}{2\left(1-e^{-\pi s}\right)}\left\{\int_0^\pi e^{-(i+s)t}dt + \int_0^\pi e^{(i-s)t}dt\right\}$$

$$\frac{1}{2\left(1-e^{-\pi s}\right)}\left\{\int_0^{\pi} e^{-(i+s)t}\,dt+\int_0^{\pi} e^{(i-s)t}\,dt\right\}=\frac{1}{2\left(1-e^{-\pi s}\right)}\left\{-\left.\frac{e^{-(i+s)t}}{(i+s)}\right|_0^{\pi}+\left.\frac{e^{(i-s)t}}{(i-s)}\right|_0^{\pi}\right\}$$

$$=\frac{1}{2\left(1-e^{-\pi s}\right)}\left\{\frac{1-e^{-i\pi}e^{-s\pi}}{(i+s)}+\frac{e^{i\pi}e^{-s\pi}-1}{(i-s)}\right\}$$

$$=\frac{1}{2\left(1-e^{-\pi s}\right)}\left\{\frac{1+e^{-s\pi}}{(i+s)}+\frac{-e^{-s\pi}-1}{(i-s)}\right\}$$

$$=\frac{1}{2\left(1-e^{-\pi s}\right)}\left\{\left[\frac{1}{i+s}-\frac{1}{i-s}\right]+e^{-s\pi}\left[\frac{1}{i+s}-\frac{1}{i-s}\right]\right\}=\frac{1}{2\left(1-e^{-\pi s}\right)}\left\{\left(1+e^{-\pi s}\right)\left(\tfrac{1}{i+s}-\tfrac{1}{i-s}\right)\right\}$$

$$=\frac{\left(1+e^{-\pi s}\right)}{2\left(1-e^{-\pi s}\right)}\left[\frac{i-s-i-s}{\left(-1-s^2\right)}\right]=\frac{\left(1+e^{-\pi s}\right)}{2\left(1-e^{-\pi s}\right)}\left[\frac{2s}{\left(s^2+1\right)}\right]=\frac{s\left(1+e^{-\pi s}\right)}{\left(1-e^{-\pi s}\right)\left(s^2+1\right)}$$

9. Use **Thm 11** to verify in the case of $\lim_{t\to\infty} f(t)$, where: $f(t) = 4te^{-3t} - e^{-3t}\sinh 2t$

$$\lim_{t\to\infty} f(t)=\lim_{t\to\infty}\left\{4te^{-3t}-e^{-3t}\sinh 2t\right\}=4\lim_{t\to\infty} te^{-3t}-\tfrac{1}{2}\left(\lim_{t\to\infty} e^{-t}-\lim_{t\to\infty} e^{-5t}\right)=0$$

$$\lim_{s\to 0} sF(s)=\lim_{s\to 0} sL\left\{4te^{-3t}-e^{-3t}\sinh 2t\right\}$$

$$L\left\{4te^{-3t}-e^{-3t}\sinh 2t\right\}=4L\left\{te^{-3t}\right\}-L\left\{e^{-3t}\sinh 2t\right\}$$

- According to THM 5: $L\left\{te^{-3t}\right\}=(-1)\frac{d}{ds}\left(e^{-3t}\right)=(-1)\frac{d}{ds}\frac{1}{s+3}=\frac{1}{(s+3)^2}$
- According to THM7: $L\left\{e^{-3t}\sinh 2t\right\}=F(s+3), F(s)=L\{\sinh 2t\}=\frac{2}{s^2-4}$
  Hence: $F(s+3)=\frac{2}{(s+3)^2-4}$

So: $L\left\{4te^{-3t}-e^{-3t}\sinh 2t\right\}=4L\left\{te^{-3t}\right\}-L\left\{e^{-3t}\sinh 2t\right\}=\frac{4}{(s+3)^2}-\frac{2}{(s+3)^2+4}$

$$\therefore \lim_{s\to 0} sF(s)=\lim_{s\to 0}\left[\frac{4s}{(s+3)^2}-\frac{2s}{(s+3)^2+4}\right]=\frac{0}{9}-\frac{0}{13}=0$$

10. Prove: $\int_{-\infty}^{\infty} f(t)\delta(t-t_0)=f(t_0)$

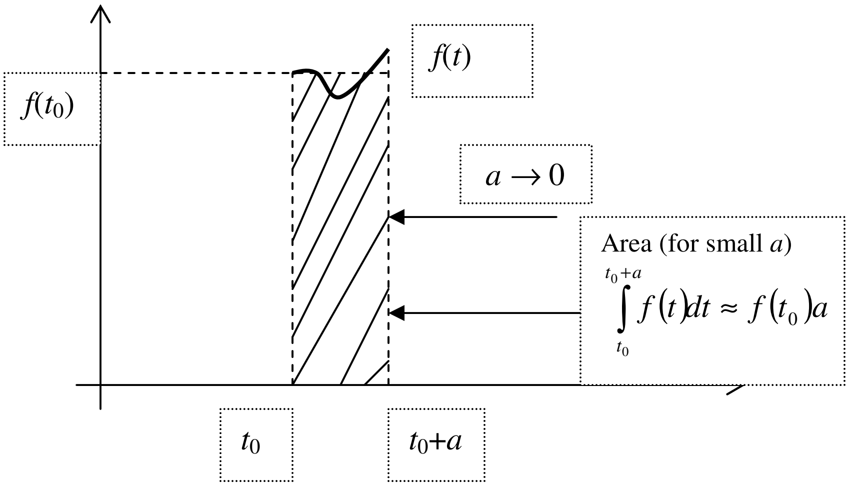
Proof:

$$\int_{-\infty}^{\infty} f(t)\delta(t-t_0) = \int_{-\infty}^{\infty} f(t)\lim_{a\to 0}\left[\frac{u(t-t_0)-u(t-t_0-a)}{a}\right]$$

$$= \lim_{a\to 0} \tfrac{1}{a}\left\{\int_{-\infty}^{\infty} f(t)u(t-t_0)dt - \int_{-\infty}^{\infty} f(t)u(t-t_0-a)dt\right\} = \lim_{a\to 0} \tfrac{1}{a}\left\{\int_{t_0}^{\infty} f(t)dt - \int_{t_0+a}^{\infty} f(t)dt\right\}$$

$$= \lim_{a\to 0} \tfrac{1}{a}\int_{t_0}^{t_0+a} f(t)dt = \lim_{a\to 0} \tfrac{1}{a}\left[f(t_0)a\right] = f(t_0)$$

The last step in the proof involved approximating the integral $\int_{t_0}^{t_0+a} f(t)dt$ with the area of a rectangle of height $f(t_0)$ and width a. Note that in the limit $a \to 0$, the approximation becomes exact. (See figure) :



11. a) Find the inverse LT of $\dfrac{1}{s(s^2+1)}$ without using Thm12

$$F(s) = \tfrac{1}{s}\tfrac{1}{s^2+1} = \tfrac{1}{s}L\{\sin t\} \Rightarrow \tfrac{1}{s}L\{\sin t\} = L\left\{\int_0^t \sin \omega dw\right\} = L\left\{-\cos\omega\big|_0^t\right\} = L\{1-\cos t\}$$

(according to THM4) So the inverse LT is (1 – cos*t*)

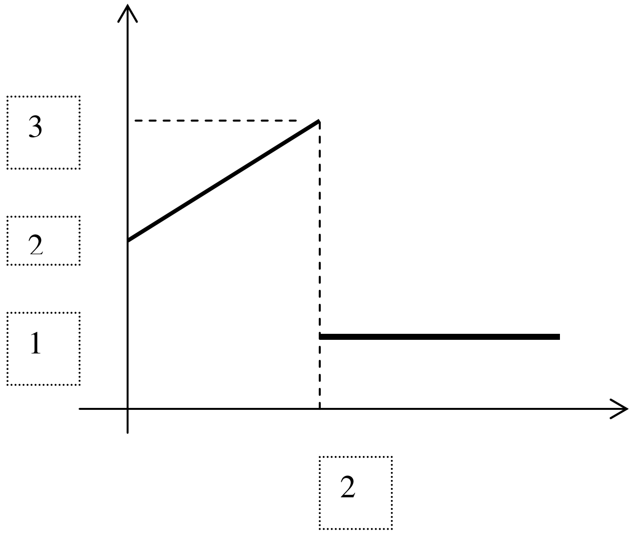
b.) Verify a) using Thm12

$$\frac{1}{s\left(s^2+1\right)}=\tfrac{1}{s}\cdot\tfrac{1}{\left(s^2+1\right)}=F(s)G(s)=L\{1\}L\{\sin t\}=L\left\{\int_0^t 1\cdot\sin(t-u)du\right\}=L\left\{\int_0^t \sin u du\right\}$$
$$=L\left\{-\cos u\Big|_0^t\right\}=L\{1-\cos t\}$$

(Notice in simplifying the integral use was made of the property of the convolution:

$\int_0^t f(u)g(t-u)du=\int_0^t f(t-u)g(u)du$, where: $f(t) = 1$, $g(t) = \sin t$)

12) Find $f(t), f'(t)$ and the associated LTs of $f(t), f'(t)$:



a.) $$f(t)=\left(\tfrac{1}{2}t+2\right)\left(u(t)-u(t-2)\right)+u(t-2)=\tfrac{1}{2}tu(t)+2u(t)-\tfrac{1}{2}tu(t-2)-u(t-2)$$
$$=\tfrac{1}{2}tu(t)+2u(t)-\tfrac{1}{2}(t-2)u(t-2)-2u(t-2)$$

b.)
$$L\{f(t)\}=\tfrac{1}{2}L\{u(t)t\}+2L\{u(t)\}-\tfrac{1}{2}L\{u(t-2)(t-2)\}-2L\{u(t-2)\}$$
$$=\tfrac{1}{2s^2}+2\tfrac{1}{s}-\tfrac{1}{2}e^{-2s}\tfrac{1}{s^2}-2\tfrac{e^{-2s}}{s}=\tfrac{1}{s^2}+\tfrac{2}{s}-\tfrac{e^{-2s}}{2s^2}-\tfrac{2e^{-2s}}{s}=\tfrac{1}{s^2}\left[\tfrac{1}{2}+2s-e^{-2s}\left(2s+\tfrac{1}{2}\right)\right]$$

Note: The first and the third LTs were obtained using THM8 (where, in the first LT, $a = 0$ and $f(t) = t$, i..e.: $e^{-0s}F(s) = L\{u(t-0)f(t-0)\}$, whereas in the third, $a = 2$ and $f(t) = t$ i.e. $e^{-2s}F(s) = L\{u(t-2)f(t-2)\}$

The first and the third LTs could just have well been obtained using THM5:

First term: $\tfrac{1}{2}L\{tu(t)\}=\tfrac{1}{2}(-1)\tfrac{d}{ds}\left(\tfrac{e^{-0s}}{s}\right)=\tfrac{1}{2}(-1)\tfrac{d}{ds}\left(\tfrac{1}{s}\right)=\tfrac{1}{2s^2}$

Third term:

$$-\tfrac{1}{2}L\{(t-2)u(t-2)\} = -\tfrac{1}{2}L\{tu(t-2)\} - \tfrac{1}{2}\cdot(-2)\cdot L\{u(t-2)\} = -\tfrac{1}{2}\cdot(-1)\tfrac{d}{ds}\left(\tfrac{e^{-2s}}{s}\right) + \left(\tfrac{e^{-2s}}{s}\right)$$
$$= \tfrac{-2se^{-2s}-e^{-2s}}{2s^2} + \tfrac{e^{-2s}}{s} = \tfrac{-2se^{-2s}-e^{-2s}}{s^2} + \tfrac{2se^{-2s}}{2s^2} = -\tfrac{e^{-2s}}{2s^2}$$

…It's obviously much easier to use the THM8 approach!

c.)

$$f'(t) = \tfrac{d}{dt}\{\tfrac{1}{2}tu(t) + 2u(t) - \tfrac{1}{2}(t-2)u(t-2) - 2u(t-2)\}$$
$$= \tfrac{1}{2}(t\delta(t) + u(t)) + 2\delta(t) - \tfrac{1}{2}((t-2)\delta(t-2) + u(t-2)) - 2\delta(t-2)$$
$$= \tfrac{1}{2}(0 + u(t)) + 2\delta(t) - \tfrac{1}{2}(t-2)\delta(t-2) - \tfrac{1}{2}u(t-2) - 2\delta(t-2)$$
$$= \tfrac{1}{2}u(t) - \tfrac{1}{2}u(t-2) + 2\delta(t) - \tfrac{1}{2}(t-2)\delta(t-2) - 2\delta(t-2)$$

(Recall: $t\delta(t) = 0$ for all t)

d.) According to THM2: $L\{f'(t)\} = sL\{f(t)\} - f(0)$

Here: $f(0) = \tfrac{1}{2}0 + 2u(0) - \tfrac{1}{2}(0-2)u(0-2) - 2u(0-2) = 0$

(since: $u(t-a) = \begin{Bmatrix} 1 & t > a \\ 0 & t < a \end{Bmatrix}, a \geq 0$,i.e., for any nonegatiuve a, by defn (p. 45 Sheng))

Hence: $u(0) = u(0-0) = 0, u(0-2) = 0$

So: $L\{f'(t)\} = s\cdot\tfrac{1}{s^2}\left[\tfrac{1}{2} + 2s - e^{-2s}(2s + \tfrac{1}{2})\right] - f(0) = \tfrac{1}{s}\left[\tfrac{1}{2} + 2s - e^{-2s}(2s + \tfrac{1}{2})\right]$

This result can be verified by direct computation:

$$L\{f'(t)\} = L\{\tfrac{1}{2}u(t) - \tfrac{1}{2}u(t-2) + 2\delta(t) - \tfrac{1}{2}(t-2)\delta(t-2) - 2\delta(t-2)\}$$
$$= \tfrac{1}{2}L\{u(t)\} - \tfrac{1}{2}L\{u(t-2)\} + 2L\{\delta(t)\} - \tfrac{1}{2}L\{t\delta(t-2)\} - L\{\delta(t-2)\}$$
$$= \tfrac{1}{2}\tfrac{1}{s} - \tfrac{1}{2}\cdot\tfrac{e^{-2s}}{s} + 2 - \tfrac{1}{2}(-1)\tfrac{d}{ds}\left(e^{-2s}\right) - e^{-2s}$$
$$= \tfrac{1}{2s} - \tfrac{e^{-2s}}{2s} + 2 - e^{-2s} - e^{-2s} = \tfrac{1}{2s} + 2 - \tfrac{e^{-2s}}{2s} - 2e^{-2s} = \tfrac{1}{s}\left[\tfrac{1}{2} + 2s - e^{-2s}(2s + \tfrac{1}{2})\right]$$