MA360 **Handout 2a (week of Jan 16)**
Sp2006 Kallfelz
Assignment (revised) : 3.(b),(c)/4.(b),(c) (pg 18) [Read pp21-29, Sheng]
1.(a),(e),(g),(j),(k),(o)/2.(c),(d),(f),(h),(i) (pp30-31) [Read 34-40, Sheng]

In addition to the list (from Handout 2):

1) $L(e^{-at}) = \frac{1}{s-(-a)}$, 2) $L(\sin\theta t) = L\left(\frac{e^{i\theta t}-e^{-i\theta t}}{2i}\right) = \frac{\theta}{s^2+\theta^2}$, 3) $L(\cos\theta t) = \frac{s}{s^2+\theta^2}$,
4) $L(\sinh at) = L\left(\frac{e^{at}-e^{-at}}{2}\right) = \frac{a}{s^2-a^2}$, 5) $L(\cosh at) = L\left(\frac{e^{at}+e^{-at}}{2}\right) = \frac{s}{s^2-a^2}$,
6) $L\{t^n\} = \frac{\Gamma(n+1)}{s^{p+1}} = \frac{n!}{s^{p+1}}$ where n is any nonnegative integer
7) note also that: $L\{u(t-a)\} = \frac{e^{-as}}{s}$

since:

$$L[u(t-a)] = \int_0^\infty u(t-a)e^{-st}\,dt = \int_0^a 0\cdot e^{-as}\,dt + \int_a^\infty 1\cdot e^{-st}\,dt = \int_a^\infty e^{-st}\,dt$$

$$= -\tfrac{1}{s}e^{-st}\Big|_a^\infty = -\tfrac{1}{s}[\lim_{d\to\infty} e^{-sd} - e^{-sa}] = \frac{e^{-sa}}{s}$$

These are useful when seeking to obtain the inverse LT:

- <u>Example1 (similar to 2(d), Sheng, p.31):</u> Find $L^{-1}\left[\frac{s+2}{(s+5)^2+16}\right]$

Step 1: The parenthesis term in the denominator alerts to a shift (from Thm 7: $L\{f(t)\} = F(s) \Rightarrow L\{e^{at}f(t)\} = F(s-a)$, where a = -5 in this case.) Hence rewrite:

$F(s+5) = \left[\frac{s+2}{(s+5)^2+16}\right] = \left[\frac{(s+5)-3}{(s+5)^2+16}\right]$ (since *every* occurrence of s must be so adjusted)

Hence: $F(s) = \left[\frac{s-3}{s^2+16}\right] = \frac{s}{s^2+16} - \frac{3}{s^2+16}$

Step 2: From our 'toolchest' above, we recognize immediately that the first term is the LT of cos4t. The second term can be easily converted to become the LT of sin4t:

$$\frac{-3}{s^2+16} = -\frac{3}{4}\frac{4}{s^2+16} = -\frac{3}{4}L[\sin 4t]$$

So: $F(s) = \frac{s}{s^2+16} - \frac{3}{s^2+16} = L[\cos 4t] - \frac{3}{4}L[\sin 4t] = L\left[\frac{1}{4}(4\cos 4t - 3\sin 4t)\right]$

So: $F(s+5) = L\left\{e^{-5t}\cdot\frac{1}{4}[\cos 4t - 3\sin 4t]\right\}$

Therefore: $L^{-1}\left[\frac{s+2}{(s+5)^2+16}\right] = \frac{e^{-5t}}{4}[\cos 4t - 3\sin 4t]$

- <u>Example (2(e), p31, Sheng)</u>

Find: $L^{-1}\left[\frac{e^{-3s}}{s^4}\right]$

According to Thm 8:

$$F(s)=\left[\frac{e^{-3s}}{s^4}\right]=e^{-3s}\cdot\frac{1}{s^4}=L[u(t-3)f(t-3)]$$
$$where:\frac{1}{s^4}=L[f(t)]$$

Hence: $\frac{1}{s^4}=\frac{1}{3!}\frac{3!}{s^4}=\frac{1}{6}L\left[t^3\right]$ (using Formula 1c.2: $L\{t^n\}=\frac{\Gamma(n+1)}{s^{p+1}}=\frac{n!}{s^{p+1}}$ (Handout 1c))

Hence: $F(s)=e^{-3s}\cdot\frac{1}{s^4}=L[u(t-3)f(t-3)]=L\left[u(t-3)\frac{1}{6}(t-3)^3\right]$

So: $L^{-1}\left[\frac{e^{-3s}}{s^4}\right]=\frac{1}{6}u(t-3)(t-3)^3=\begin{Bmatrix}\frac{1}{6}(t-3)^3 & t>3\\ 0 & t\le 3\end{Bmatrix}=\frac{1}{6}(t-3)^3, t>3$

- <u>Example (2(g), p31, Sheng)</u>

$L^{-1}\left[\frac{e^{-\pi s}}{s^2+25}\right]$

According to Thm 8:

$$F(s)=\left[\frac{e^{-\pi s}}{s^2+25}\right]=e^{-\pi s}\cdot\frac{1}{s^2+25}=L[u(t-\pi)f(t-\pi)]$$
$$where:\frac{1}{s^2+25}=L[f(t)]$$

Hence: $\frac{1}{s^2+25}=\frac{1}{5}\frac{5}{s^2+5^2}=\frac{1}{5}L[\sin 5t]=L\left[\frac{1}{5}\sin 5t\right]$

Hence: $L^{-1}\left[\frac{e^{-\pi s}}{s^2+25}\right]=\frac{1}{5}u(t-\pi)\sin 5(t-\pi)=\begin{Bmatrix}\frac{1}{5}\sin 5(t-\pi) & t>\pi\\ 0 & t\le\pi\end{Bmatrix}=\frac{1}{5}\sin 5(t-\pi), t>\pi$

Moreover: $\sin 5(t-\pi)=\sin(5t-5\pi)=\sin 5t\cos 5\pi-\cos 5t\sin 5\pi=-\sin 5t$

So: $L^{-1}\left[\frac{e^{-\pi s}}{s^2+25}\right]=-\frac{1}{5}\sin 5t, t>\pi$