MA360 Handout 2 (week of Jan 16)
Sp2006 Kallfelz
Assignment: 3.(b),(d)/4.(b),(c) (pg 18) [Read pp21-29, Sheng]
1.(b),(f),(g),(j),(l),(n)/2.(c),(d),(g),(h),(i) (pp30-31) [Read 34-40, Sheng]

You've seen some applications of Thms 1-5, Sheng, as well as (in Handout 1c) how to apply some of them, to avoid resorting to calculating to LT from first principles alone (mostly what was shown in Handout 1b). As I emphasized last week, though, all the shortcut theorems presented in chapter I Sheng are not meant to *substitute* what you know about the techniques and concepts behind complicated integrals (just as shortcut theorems in Calc I,II aren't meant to *substitute* what you know about the techniques and concepts behind the evaluation of limits), but to *supplement* what you know, and can therefore apply, as a practical labor-saving device. In addition (as bonus material) I introduced the Gamma function (last pages, Handout 1c) as a means of directly evaluating the LP for functions of the form: $f(t) = t^p$ (for $p > -1$) (an extension to <u>Example 1-2-4</u>, pg. 9)

To invoke the Calculus analogy again, the Laplace Transform is an *operator*, i.e., a mathematical object that acts on *functions* (the way functions act on *variables*). Single-variable Calculus (Calculus on functions of the form $f(x)$) is basically a course in which you became thoroughly acquainted with the derivative *operator* $D_t\{f(t)\} \equiv {}^{df}/_{dt}$(and its associated inverse operator[1]: $D_t^{-1}\{f(t)\} \equiv \int f(t)dt$, i.e. the 'anti-derivative'). (Multivariate calculus is just a generalization of what you knew about the derivative operator, and its inverse, applied to vector-valued functions $f: R^n \to R^m$, or functions that map an array of n real numbers to another array of m real numbers[2].) The point is, the focus is on this new *operator*. How did you learn about it? You were first introduced to the *derivative* D_t to the point that you 'got used to' expressions of the derivatives for simple functions, so that you could naturally 'go backwards' in the inverse process, i.e., take their integrals. The same game is at work here: you're getting accustomed to taking the LT's for simple functions, to the point that you can naturally 'go backwards' and find their associated *inverse* LT's : (i.e. $L\{f(t)\} = F(s)$, so $L^{-1}\{F(s)\} = f(t)$). This informal approach is the best way to get a 'working knowledge' of the LT before getting more formal about it.

We'll summarize first some of the theorems (next page)

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[1] Just as some *functions* are invertible (i.e. possess an *inverse,* i.e., a function f^{-1} such that: $(f \circ f^{-1})(x) = x = (f^{-1} \circ f)(x)$), likewise some operators $\mathbf{O}$ are invertible $(\mathbf{O} \circ \mathbf{O}^{-1})\{f(x)\} = x = (\mathbf{O}^{-1} \circ \mathbf{O})\{f(x)\}$. The derivative operator and the Laplace Transform operator are examples of these.

[2] Of course, subtleties arise in the case of the operators themselves, when generalized over an $n \times m$ – dimensional real-valued vector space. (For instance, the derivate operator D, depending on the context, can become the *gradient* operator, or the *Jacobian*, etc., all necessarily involving the concept of partial derivatives and, in the inverse case, multivariate integration.)

Theorems	**Description**
1. $L\{af(t) + bg(t)\} = aL\{f(t)\} + bL\{g(t)\}$	Linearity property of LT
2 .: $L\{D_t f(t)\} = sL\{f(t)\} - f(0)$ 2a. Corrolary to 2 (not found in Sheng): $L\{D_t^{\,n} f(t)\} = s^n L\{f(t)\} - \sum^{n}_{j=1} s^{n-j} f^{(j-1)}(0)$ 4 : $L\{D_t^{\,-1} f(t)\} = {}^1/s L\{f(t)\}$	2. & 4. give formulae in terms of the LT of f (i.e., $L\{f(t)\} = F(s)$) when you're faced with computing the LT for the derivative of f (or in the case of Thm 2a, the n-th derivative) or the antiderivative of f $(D_t^{-1}\{f(t)\} \equiv \int^t_0 f(\omega)d\omega$
3. $L\{f(t)\} = F(s) \Rightarrow L\{f(at)\} = {}^1/_a F({}^s/_a)$	An obviously useful simplification formula, but it also tells you that the LT is *not* linear with respect to the *variable* arguments *of f*. LT *is* linear with respect to *its functional* arguments[3]!
5. $L\{t^n f(t)\} = (-1)^n D^n_s L\{f(t)\}$ for $n = 0,1,2,\ldots$	Again, another obviously useful simplification formula
5a. Corrolary (Example 1-2-4) $L\{t^p\} = \frac{\Gamma(p+1)}{s^{p+1}}$, for *any* $p > -1$	(From Handout 1c, Formula 1c.1) The formula comes especially in handy of integer and half-integer exponents, given Properties 1-4 as of the Gamma function as summarized in Handout 1c
6. $L\{t^{-1} f(t)\} = \int_s^\infty F(\omega)d\omega$ where: $L\{f(t)\} = F(s)$	Discussed here (Handout 2). Calculus analogy: Recall the power-rule for integration did *not* hold for t^p when $p = -1$? (You had to introduce a rigorous defn of logarithm)
7. $L\{f(t)\} = F(s) \Rightarrow L\{e^{at} f(t)\} = F(s - a)$	Shifting Thm 1 (discussed here)
8. $L\{f(t)\} = F(s) \Rightarrow$ $L\{u(t-a)f(t-a)\} = e^{-as}F(s)$	Shifting Thm2 (discussed here)

- **Proof of Corrolary 2a (by Induction)**

<u>Base Case:</u> (Proved in the case n =1, i.e. the proof of Thm 2, page 8 Sheng)

<u>Induction Case:</u> Assume: $L\left[D_t^k f(t)\right] = s^k L[f(t)] + \sum_{j=1}^{k} s^{k-j} f^{(j-1)}(0)$ for any $k = 2,3,\ldots.$

Prove: $L\left[D_t^{k+1} f(t)\right] = s^{k+1} L[f(t)] + \sum_{j=1}^{k+1} s^{k+1-j} f^{(j-1)}(0) = s^{k+1} L[f(t)] + \sum_{j=0}^{k+1} s^{k-j} f^{(j-1)}(0)$

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[3] I.e., as Thm1 states: $L\{af(t) + bg(t)\} = aL\{f(t)\} + bL\{g(t)\}$.
However: $L\{f(at + bt))\} \neq aL\{f(t)\} + bL\{f(t))\}$. This tells us LT *is* a linear *operator* (it's linear with respect to *functions*, which are its *arguments*), but it is certainly *not* a linear *function* (it's not linear with respect to the function's *variable* arguments.)

Proof: $L\left[D_t^{k+1} f(t)\right] = L\left[D_t D_t^k f(t)\right]$

Re-name: $F(t) = D_t^k f(t)$

Then according to Thm2 (proved in base case):

$$L[D_t F(t)] = sL[F(t)] - F(0)$$

Substituting:

$$L\left[D_t^{k+1} f(t)\right] = s\left\{s^k L[f(t)] - \sum_{j=1}^{k} s^{k-j} f^{(j-1)}(0)\right\} - \left[D_t^k f(0)\right]$$
$$= s^{k+1} L[f(t)] - s\left\{s^{k-1} f(0) + s^{k-2} f'(0) + \ldots + s^2 f^{(k-3)}(0) + s f^{(k-2)}(0) + f^{(k-1)}(0)\right\} - f^{(k)}(0)$$
$$= s^{k+1} L[f(t)] - \left\{s^k f(0) + s^{k-1} f'(0) + \ldots + s^2 f^{(k-2)}(0) + s f^{(k-1)}(0) + f^{(k)}(0)\right\}$$
$$= s^{k+1} L[f(t)] - \sum_{j=0}^{k+1} s^{k+1-j} f^{(j-1)}$$

Hence, for any positive integer n : $L\left[D_t^n f(t)\right] = s^n L[f(t)] + \sum_{j=1}^{n} s^{n-j} f^{(j-1)}(0)$

- **Another way to Prove Thm6**

Sheng proves Thm6 directly, by iterated integration. Another way is to use differentiation and Thm 5:

Let $g(t) = {}^{f(t)}/_{t}$, then: $f(t) = tg(t)$. Hence:

$$L\{f(t)\} = L\{tg(t)\} = (-1)^1 D^n{}_s L\{g(t)\}$$

(using Thm 5, for $n = 1$), i.e.:

$$(-1) D_s L[g(t)] = -\tfrac{d}{ds} L[g(t)] = -\tfrac{d}{ds} L\left[\tfrac{f(t)}{t}\right] = L[tg(t)] = L[f(t)] = F(s)$$

Hence: $-\tfrac{d}{ds} L\left[\tfrac{f(t)}{t}\right] = F(s) \Rightarrow L\left[\tfrac{f(t)}{t}\right] = -\int_c^s F(\omega)d\omega = \int_s^c F(\omega)d\omega$

Note1: for some constant c, using the fundamental theorem of calculus and the property that reversing the order of the integration changes its sign)

However, a central property of the LT is $\lim_{s\to\infty} F(s) = 0$, if f is piecewise continuous[4]

Hence: $-\tfrac{d}{ds} L\left[\tfrac{f(t)}{t}\right] = F(s) \Rightarrow L\left[\tfrac{f(t)}{t}\right] = \lim_{c\to\infty} \int_s^c F(\omega)d\omega = \int_s^\infty F(\omega)d\omega$

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[4] A bonus exercise in Assignment I

- **The Shifting Theorems**

The proof of Thm 7 (p 22 Sheng) and the idea behind it is straightforward. Carefully look over the discussion on the step function and Thm8, however, as it's a little more subtle. (pp 24 – 25)

Note also that there's a minor error in Sheng's Proof: (page 25)

The second integral (read down from the top of page 25) should read:

$$\int_a^\infty e^{-s\omega} f(\omega - a)\,d\omega = e^{-as}F(s)$$

i.e., the bottom limit of integration should = *a*, not 0, since in the change-of-variable substitution: $\omega = t + a$, we must adjust the limits of integration as well (i.e., recall from the *u* – substitution in definite integration: $\int_a^b f(x)dx = \int_{u(a)}^{u(b)} f(u)du$.)

The reason why the error is minor, however, is because:

$$\int_a^\infty e^{-s\omega} f(t-a)dt = \int_o^\omega u(t-a)f(t-a)dt = e^{-as}F(s)$$

because of the property of the step-function. I.e., for any function *g*:

$$\int_0^\infty u(t-a)g(t)dt = \int_0^a u(t-a)g(t)dt + \int_a^\infty u(t-a)g(t)dt = \int_0^a 0\cdot g(t)dt + \int_a^\infty 1\cdot g(t)dt = \int_a^\infty g(t)dt$$

- **Discussion and Examples**

At this point, you've built up a little 'tool-box' composed of LT's of some elementary functions:

$L(e^{-at}) = \frac{1}{s-(-a)}$, $L(\sin\theta t) = L\left(\frac{e^{i\theta t}-e^{-i\theta t}}{2i}\right) = \frac{\theta}{s^2+\theta^2}$, $L(\cos\theta t) = \frac{s}{s^2+\theta^2}$,
$L(\sinh at) = L\left(\frac{e^{at}-e^{-at}}{2}\right) = \frac{a}{s^2-a^2}$, $L(\cosh at) = L\left(\frac{e^{at}+e^{-at}}{2}\right) = \frac{s}{s^2-a^2}$,
$L\{t^n\} = \frac{\Gamma(n+1)}{s^{p+1}} = \frac{n!}{s^{p+1}}$ where *n* is any nonnegative integer
(the last is Formula 1c.2, from Handout 1c)

Keeping the above in mind, and remark I made about finding the inverse transforms (by getting 'used to' the LTs of such functions to the point where you can reason backwards, similar to when you were first introduced to integration) consider the examples below:

3.(a) (p 18, Sheng)

Find: $L^{-1}\left[\frac{1}{s(s-2)}\right] = L^{-1}[F(s)]$

First of all, from Thm4 and from the above expressions, note:

$$F(s) = \frac{1}{s(s-2)} = \frac{1}{s}G(s) \Rightarrow G(s) = \frac{1}{s-2} = L\left[e^{2t}\right]$$

According to Thm4: $L\{D_t^{-1}f(t)\} = {}^{1}/s\,L\{f(t)\}$

Hence: (since $f(t) = e^{2t}$, as determined above)

$$F(s) = \tfrac{1}{s}L\{f(t)\} = L\left[D_t^{-1}f(t)\right] = L\left[\int_0^s f(\omega)d\omega\right] \Rightarrow$$

$$F(s) = L\left[\int_0^s e^{2t}dt\right] = L\left[\tfrac{1}{2}e\Big|_0^s\right] = L\left[\tfrac{1}{2}e^{2s} - \tfrac{1}{2}e^0\right] = L\left[\tfrac{1}{2}\left(e^{2t} - 1\right)\right]$$

$$\therefore L^{-1}[F(s)] = \tfrac{1}{2}\left(e^{2t} - 1\right)$$

3.(d) (p 18, Sheng)

Find: $L^{-1}\left[\frac{1}{s\left(s^2+16\right)}\right] = L^{-1}[F(s)]$

$$F(s) = \frac{1}{s\left(s^2+16\right)} = \frac{1}{s}G(s) \Rightarrow G(s) = \frac{1}{4}\frac{4}{s+16} = \frac{1}{4}L[\sin 4t]$$

Hence: (since $f(t) = \sin 4t$, as determined above)

$$F(s) = \tfrac{1}{4}\tfrac{1}{s}L\{f(t)\} = \tfrac{1}{4}L\left[D_t^{-1}f(t)\right] = \tfrac{1}{4}L\left[\int_0^s f(\omega)d\omega\right] \Rightarrow$$

$$F(s) = \tfrac{1}{4}L\left[\int_0^s \sin 4t\,dt\right] = \tfrac{1}{4}L\left[-\tfrac{1}{4}\cos 4t\Big|_0^s\right] = -\tfrac{1}{4}L\left[\tfrac{1}{2}\cos 4s - \tfrac{1}{2}\cos 0\right] = -\tfrac{1}{4}L\left[\tfrac{1}{4}(\cos 4s - 1)\right]$$

$$= L\left[-\tfrac{1}{16}(\cos 4s - 1)\right] = L\left[\tfrac{1}{16}(1 - \cos 4s)\right]$$

$$\therefore L^{-1}[F(s)] = \tfrac{1}{16}(1 - \cos 4s)$$

Additional Examples:

1.(b) (p 30, Sheng)

Find: $L\left[\frac{1}{t}\left(e^{-2t}-e^{-3t}\right)\right]=L\left[\frac{e^{-2t}}{t}\right]-L\left[\frac{e^{-3t}}{t}\right]$

According to Thm 6: $L\left\{t^{-1}f(t)\right\}=\int_s^\infty F(\omega)d\omega$ where: $L\{f(t)\} = F(s)$

Hence:

$$L\left[\frac{e^{-2t}}{t}\right]-L\left[\frac{e^{-3}}{t}\right]=\int_s^\infty L\left[e^{-2t}\right]d\omega-\int_s^\infty L\left[e^{-3t}\right]d\omega$$

$$=\int_s^\infty \frac{d\omega}{\omega+2}-\int_s^\infty \frac{d\omega}{\omega+3}=\lim_{d\to\infty}\left\{\ln|\omega+2|\Big|_s^d-\ln|\omega+3|\Big|_s^d\right\}$$

$$=\lim_{d\to\infty}\ln\left|\frac{\omega+2}{\omega+3}\right|\Big|_s^d=\lim_{d\to\infty}\ln\left|\frac{d+2}{d+3}\right|-\ln\left|\frac{s+2}{s+3}\right|$$

$$=\lim_{d\to\infty}\ln\left|\frac{1+\frac{2}{d}}{1+\frac{3}{d}}\right|-\ln\left|\frac{s+2}{s+3}\right|=\ln\left|\frac{1+0}{1+0}\right|+\ln\left(\left|\frac{s+2}{s+3}\right|^{-1}\right)$$

$$=\ln 1+\ln\left|\frac{s+3}{s+2}\right|=\ln\left|\frac{s+3}{s+2}\right|$$

<u>1.(f) (p 30, Sheng)</u>

Find: $L\left[e^{-6t}(3\cos 2t-5\sin 2t)\right]=3L\left[e^{-6t}\cos 2t\right]-5L\left[e^{-6t}\sin 2t\right]$

According to Thm7: $L[e^{-6t}\cos 2t] = F(s + 6)$, where:

$$F(s)=L(\cos\theta t)=\frac{s}{s^2+\theta^2}\Rightarrow L[\cos 2t]=\frac{s}{s^2+4}$$

$L[e^{-6t}\sin 2t] = F(s + 6)$, where:

$$F(s)=L(\sin\theta t)=\frac{\theta}{s^2+\theta^2}\Rightarrow L[\sin 2t]=\frac{2}{s^2+4}$$

So:

$$L\left[e^{-6t}(3\cos 2t-5\sin 2t)\right]=3L\left[e^{-6t}\cos 2t\right]-5L\left[e^{-6t}\sin 2t\right]$$

$$=3\left[\frac{(s+6)}{(s+6)^2+4}\right]-5\left[\frac{2}{(s+6)^2+4}\right]=\frac{3s+8}{(s+6)^2+4}$$

<u>1.(l) (p 30, Sheng)</u>

Find: $L\left[t\left(e^{-t}\cos 2t\right)\right]$

According to Thm 5: $L\{t^n f(t)\} = (-1)^n D^n{}_s L\{f(t)\}$ for $n = 0,1,2,\ldots$

Hence, for $n = 1$: $L\left[t\left(e^{-t}\cos 2t\right)\right]=-\frac{d}{ds}L\left[e^{-t}\cos 2t\right]$

According to Thm 7: $L[e^{-t}\cos 2t] = F(s + 1)$, where:

$$F(s)=L(\cos\theta t)=\frac{s}{s^2+\theta^2}\Rightarrow L[\cos 2t]=\frac{s}{s^2+4}$$

so: $F(s+1)=\frac{s+1}{(s+1)^2+4}$

Hence:

$$L\left[t\left(e^{-t}\cos 2t\right)\right]=-\frac{d}{ds}L\left[e^{-t}\cos 2t\right]=-\frac{d}{ds}\left[\frac{s+1}{(s+1)^2+4}\right]=$$

$$-\frac{(s+1)^2+4-(s+1)[2(s+1)]}{\left[(s+1)^2+4\right]^2}=-\frac{-(s+1)^2+4}{\left[(s+1)^2+4\right]}=\frac{(s+1)^2-4}{\left[(s+1)^2+4\right]}$$

1.(n) (p 30, Sheng)

Find: $L\left[\int_0^t e^{-6\omega}\cos 8\omega d\omega\right]$

According to Thm4: $L\left[\int_0^t e^{-6\omega}\cos 8\omega d\omega\right]=\frac{1}{s}L\left[e^{-6t}\cos 8t\right]$

According to Thm 7: $L[e^{-6t}\cos 8t] = F(s+6)$, where:

$$F(s)=L(\cos\theta t)=\frac{s}{s^2+\theta^2}\Rightarrow L[\cos 8t]=\frac{s}{s^2+64}$$

So: $F(s+6)=\frac{s+6}{(s+6)^2+64}$

Hence: $L\left[\int_0^t e^{-6\omega}\cos 8\omega d\omega\right]=\frac{1}{s}\frac{(s+6)}{(s+6)^2+64}=\frac{s+6}{s\left[(s+6)^2+64\right]}$