MA360 Handout 1c (week of Jan 9) Addendum to Handout 1c
Sp2006 Kallfelz

- Example (Problem 2(b), Exercise 2, Sheng)

Given $L\{\cos t\}=\frac{s}{s^2+1}, L\{e^t\}=\frac{1}{s-1}$ Find $L\{3t(\cos\ 2t-e^{4t})\}$

$$L\{3t(\cos 2t-e^{4t})\}=3L\{t\cos 2t\}-3L\{te^{4t}\}=3(-1)D_sL\{\cos 2t\}-3(-1)D_sL\{e^{4t}\}$$

(Using Thm 5)

Furthermore, using Thm 3:

$$L\{\cos 2t\}=\tfrac{1}{2}F\left(\tfrac{s}{2}\right), \text{ where: } F(s)\equiv L\{\cos t\}=\frac{s}{s^2+1}$$
$$L\{e^{4t}\}=\tfrac{1}{4}G\left(\tfrac{s}{4}\right), \text{ where: } G(s)\equiv L\{e^t\}=\frac{1}{s-1}$$

Hence: $L\{\cos 2t\}=\frac{1}{2}F\left(\frac{s}{2}\right)=\dfrac{\frac{s}{2}}{\frac{s^2}{4}+1}=\frac{1}{2}\frac{2s}{s^2+4}=\frac{s}{s^2+4}$

$$L\{e^{4t}\}=\tfrac{1}{4}G\left(\tfrac{s}{4}\right)=\frac{1}{4}\frac{1}{\frac{s}{4}-1}=\frac{1}{4}\frac{4}{s-4}=\frac{1}{s-4}$$

So:

$$L\{3t(\cos 2t-e^{4t})\}=3L\{t\cos 2t\}-3L\{te^{4t}\}=3(-1)D_sL\{\cos 2t\}-3(-1)D_sL\{e^{4t}\}$$
$$-3\left\{\frac{d}{ds}\left[\frac{s}{s^2+4}\right]-\frac{d}{ds}\left[\frac{1}{s-4}\right]\right\}=-3\left\{\frac{s^2+4-2s^2}{\left(s^2+4\right)^2}+\frac{1}{(s-4)^2}\right\}$$

(using the quotient rule on the first term)

Getting a common denominator, the numerator term becomes, upon cross-multiplying and simplifying:

$$\left(-s^2+4\right)(s-4)^2+\left(s^2+4\right)^2=\left(4-s^2\right)\left(s^2-8s+16\right)+\left(s^4+8s^2+16\right)$$
$$=4s^2-32s+64-s^4+8s^3-16s^2+s^4+8s^2+16=8s^3-4s^2-32s+80$$
$$=4\left(2s^3-s^2-8s+20\right)$$

Hence: $L\{3t(\cos 2t-e^{4t})\}=-3\left\{\frac{s^2+4-2s^2}{\left(s^2+4\right)^2}+\frac{1}{(s-4)^2}\right\}=\frac{-12\left(2s^3-s^2-8s+20\right)}{\left(s^2+4\right)^2(s-4)^2}$

The Gamma Function

You have seen (in Handout 1b) how to obtain $L\{t^n\}$ by iterating Thm2 (Sheng) (Exampl 1-2-4, page 9 is an instance for obtaining $L\{t^2\}$. But, there are two questions you may be asking:

1. Is there a quicker way to do this (i.e., without having to iterate the formula in Thm2 an *n* number of times?)
2. What about generalizing to *any* exponent *p*?

There is an answer to both 1.,2.…introducing the Gamma Function!

- Defn.: The Gamma Function $\Gamma(p + 1)$ for $p > -1$ is defined as:

$$\Gamma(p+1)=\int_0^{\infty} x^p e^{-x}\,dx$$

- <u>Property 1: $\Gamma(1) = 1$</u>

Proof: For $\Gamma(1) = 1$, then $p = 0$

$$\Gamma(1)=\int_0^{\infty} x^0 e^{-x}\,dx=-\lim_{d\to\infty} e^{-x}\Big|_0^d=-\left[\lim_{d\to\infty} e^{-d}-e^0\right]=-[0-1]=1$$

- <u>Property 2: $\Gamma(1/2) = \sqrt{\pi}$</u>

Proof: For $\Gamma(1/2) = 1$, then $p = -{}^1\!/_2$

$$\Gamma\left(\tfrac{1}{2}\right)=\int_0^{\infty} x^{-\frac{1}{2}} e^{-x}\,dx \Rightarrow u=x^{\frac{1}{2}},du=\tfrac{1}{2}x^{-\frac{1}{2}}dx, x=u^2 \Rightarrow dx=2x^{\frac{1}{2}}du=2udu$$

$$\therefore \int_0^{\infty} x^{-\frac{1}{2}} e^{-x}\,dx=\int_{u(0)}^{u(\infty)} u^{-1}e^{-u^2}(2udu)=2\int_0^{\infty} e^{-u^2}du$$

This last integral is evaluated by performing the following trick:

$$\Gamma\left(\tfrac{1}{2}\right)=2\int_0^{\infty} e^{-u^2}du \Rightarrow \left[\Gamma\left(\tfrac{1}{2}\right)\right]^2=4\int_0^{\infty}\int_0^{\infty} e^{-u^2}e^{-w^2}dudw=\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\left(u^2+w^2\right)}dudw$$

(I.e. by squaring the integral we convert it into a double integral over the first quadrant of the Cartesian plane, which is one-fourth the (infinite) region of the entire Cartesian Plane.) The integral can be evaluated using a change of variables from Cartesian (u,w) to Polar Coordinates (r,θ):

$$\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} e^{-\left(u^2+w^2\right)}dudw=\int_0^{2\pi}\int^{\infty} e^{-r^2}rdrd\theta=\int_0^{2\pi}d\theta\int_0^{\infty} e^{-r^2}rdr=2\pi\left\{-\tfrac{1}{2}e^{-r^2}\Big|_0^{\infty}\right\}$$

(The integral with the exponential term is evaluated using the u – subsitution: $u = r^2$)

Hence:

$$\left[\Gamma\left(\tfrac{1}{2}\right)\right]^2 = 2\pi\left\{-\tfrac{1}{2}\left[\lim_{d\to\infty} e^{-d^2} - e^0\right]\right\} = -\pi[0-1] = \pi$$
$$\therefore \Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}$$

- <u>Property 3: $\Gamma(p+1) = p\Gamma(p)$</u>

  Proof: $\Gamma(p+1) = \int_0^\infty x^p e^{-x}dx$

  Integrating by Parts: $U = x^p$, $dV = e^{-x}$. Hence: $dU = px^{p-1}$, $V = -e^{-x}$

  Hence:
$$\Gamma(p+1) = \int_0^\infty x^p e^{-x}dx = \lim_{d\to\infty} - e^{-x}x^p\Big|_0^d - (-p\int_0^\infty x^{p-1}e^{-x}dx)$$
$$= \lim_{d\to\infty} - e^{-d}d^p - -0\cdot e^0 + p\int_0^\infty x^{p-1}e^{-x}dx = 0 + p\int_0^\infty x^{p-1}e^{-x}dx$$

  (Recall remark in Handouts 1a, 1b: $\lim_{x\to\infty}(x^n e^{-x}) = 0$, for *any n*, hence the first limit term =0.)

  So we derived: $\Gamma(p+1) = p\int_0^\infty x^{p-1}e^{-x}dx$

  But by inspection (according to the definition of the Gamma function):

$$\Gamma(p) = \int_0^\infty x^{p-1}e^{-x}dx$$

  Hence: $\Gamma(p+1) = p\int_0^\infty x^{p-1}e^{-x}dx = p\Gamma(p)$

- <u>Property 4: $\Gamma(n) = (n-1)!$, for any nonzero integer n, where ! is the factorial.</u>

  Proof: By Property 3: $\Gamma(2) = 1\cdot\Gamma(1)$
  But according to Property 1: $\Gamma(1) = 1$, Hence $\Gamma(2) = 1\cdot1 = 1$

  Iterating: $\Gamma(3) = 2\cdot\Gamma(2) = 2\cdot1 = 2! = (3-1)!$
  $\Gamma(4) = 3\cdot\Gamma(3) = 3\cdot2\cdot1 = 3! = (4-1)!$
  Etc…
  Therefore: $\Gamma(n) = (n-1)!$

What Property 4 tells us in essence is that the Gamma Function is a generalization of the factorial function. The factorial function $n!$ is defined for all non-negative integers n. The Gamma function extends the factorial into cases of p, where $p > -1$, but p can be any non-integer quantity.

Note the similar *form* in the definitions of the Laplace Transform and the Gamma Function! Here's how they're related:

Lemma 1: $L\{t^p\} = \dfrac{\Gamma(p+1)}{s^{p+1}}$**, for any** $p > -1$ **(Formula 1c.1)**

Proof: $L\{t^p\} = \int_0^\infty t^p e^{-st}\,dt$ Let $u = st$, hence: $t = \frac{u}{s}, dt = \frac{1}{s}du$

So: $L\{t^p\} = \int_0^\infty t^p e^{-st}\,dt = \int_0^\infty \left(\frac{u}{s}\right)^p e^{-u}\left(\frac{du}{s}\right) = \frac{1}{s^{p+1}}\int_0^\infty u^p e^{-u}\,du = \frac{1}{s^{p+1}}\Gamma(p+1)$

(**Note1:** s behaves like a constant in the above integral.)

So we answered our second question! There *is* an elegant way to express the LT for any power form t^p, provided $p > -1$. We can answer our first question in the special case when $p = n$, where n is a non-negative integer:

According to Lemma 1: $L\{t^n\} = \dfrac{\Gamma(n+1)}{s^{p+1}}$

But according to Property 4: $\Gamma(n + 1) = (n+1-1)! = n!$

$$\boxed{\text{Therefore: } L\{t^n\} = \frac{\Gamma(n+1)}{s^{p+1}} = \frac{n!}{s^{p+1}}}$$

(Formula 1c.2)

- Example: Find $L\{2t^6 - 5t^{5/2} + 2\}$

$L\{2t^6 - 5t^{5/2} + 2\} = 2L\{t^6\} - 5L\{t^{5/2}\} + 2L\{1\}$

According to Formula 1c.2: $L\{t^6\} = \dfrac{6!}{s^{6+1}} = \frac{6!}{s^7} = \frac{720}{s^7}$

According to Formula 1c.1: $L\{t^{5/2}\} = \dfrac{\Gamma\left(\frac{5}{2}+1\right)}{s^{\frac{5}{2}+1}} = s^{-\frac{7}{2}}\Gamma\left(\frac{7}{2}\right)$

We can use Property3 and Property2 to simplify: $\Gamma\left(\frac{7}{2}\right)$**:**

According to Property 3: $\Gamma\left(\frac{7}{2}\right)=\Gamma\left(\frac{5}{2}+1\right)=\frac{5}{2}\Gamma\left(\frac{5}{2}\right)$,

$\Gamma\left(\frac{5}{2}\right)=\Gamma\left(\frac{3}{2}+1\right)=\frac{3}{2}\Gamma\left(\frac{3}{2}\right)$,

$\Gamma\left(\frac{3}{2}\right)=\Gamma\left(\frac{1}{2}+1\right)=\frac{1}{2}\Gamma\left(\frac{1}{2}\right)$.

According to Property 2: $\Gamma\left(\frac{1}{2}\right)=\sqrt{\pi}$

Hence: $\Gamma\left(\frac{7}{2}\right)=\frac{5\cdot3\cdot1}{2^3}\sqrt{\pi}=\frac{15}{8}\sqrt{\pi}$

So: $L\left\{t^{5/2}\right\}=s^{-\frac{7}{2}}\Gamma\left(\frac{7}{2}\right)=s^{-\frac{7}{2}}\frac{15}{8}\sqrt{\pi}=\frac{15}{8s^3}\sqrt{\frac{1}{\pi s}}=\frac{15\sqrt{\pi s}}{8\pi s^4}$ (rationalizing the denominator)

Hence: $L\{2t^6-5t^{5/2}+2\}=2L\{t^6\}-5L\{t^{5/2}\}+2L\{1\}$

$$=\frac{1440}{s^7}-\frac{75\sqrt{\pi s}}{8\pi s^4}+\frac{2}{s}$$

(The last result: $L\{1\}$ of course is easily obtained in the same manner:

$$L\{1\}=L\left\{t^0\right\}=\frac{0!}{s^{0+1}}=\frac{1}{s}$$