

This cursory review of linear ODEs (first and second order) is entirely example-driven.

- **First-order**

Any garden variety first order linear ODE is of the form: $a_0(t)\dot{x}(t) + a_1(t)x(t) = f(t)$, where: $\dot{x}(t) \equiv \frac{d}{dt} x(t)$, for the unknown function $x(t)$ and $a_0(t), a_1(t), f(t)$ are given functions. We can always make this ODE simpler (by setting the leading term to be *monic*, i.e. the leading coefficient = 1) by dividing through by $a_0(t)$.¹ Hence:

$$\dot{x}(t) + p(t)x(t) = g(t), \text{ where: } p(t) = \frac{a_1(t)}{a_0(t)}, g(t) = \frac{f(t)}{a_0(t)}$$

Lemma1: Multiplying (Eqn1) by the integrating factor $r(t) = e^{\int p(t)dt}$ makes it separable.

Proof: Multiply both sides by $r(t)$ and bringing all terms to one side:

$$[p(t)x(t) - g(t)]r(t) + r(t)\dot{x}(t) = 0$$

If the above is separable, then the following condition must hold²:

$$\frac{\partial}{\partial x} \{ [p(t)x(t) - g(t)]r(t) \} = \frac{\partial}{\partial t} r(t)$$

For the left hand side: $\frac{\partial}{\partial x} \{ [p(t)x(t) - g(t)]r(t) \} = \frac{\partial}{\partial x} (p(t)x(t)r(t)) - \frac{\partial}{\partial x} (g(t)r(t))$

Since p, g, r are all homogeneous functions depending on t only, they behave as constants, with respect to taking the partial derivative with respect to x . Hence the left side simplifies to:

$$p(t)r(t) \quad \text{(Since: } \frac{\partial}{\partial x} p(t)x(t)r(t) = p(t)r(t)\frac{\partial}{\partial x} x = p(t)r(t) \text{ and } \frac{\partial}{\partial x} g(t)r(t) = 0 \text{)}$$

On the other hand, the right hand side is just a total derivative with respect to t , since r is a homogeneous function with respect to t : $\frac{\partial}{\partial t} r(t) = \frac{d}{dt} r(t)$

So the above simplifies to: $p(t)r(t) = \frac{d}{dt} r(t)$, which implies that $r(t) = e^{\int p(t)dt}$.

¹ Since the leading coefficient function must be nonzero.

² This is true because if we assume there exists an exact solution curve $\phi(x,t) = c$ (in the (x, t) plane) for the equation: $M(x,t) + N(x,t)x'(t) = 0$, then, when taking the total differential of $\phi(x,t) = c$:

$d\phi(x,t) = 0 = \frac{\partial\phi}{\partial t} dt + \frac{\partial\phi}{\partial x} dx \Rightarrow 0 = \frac{\partial\phi}{\partial t} + \frac{\partial\phi}{\partial x} \frac{dx}{dt} = M(x,t) + N(x,t)x'(t) \Rightarrow M(x,t) = -\frac{\partial\phi}{\partial t}, N(x,t) = \frac{\partial\phi}{\partial x}$.

So because of the continuity of $\phi(x,t)$: $\frac{\partial^2\phi}{\partial t\partial x} = \frac{\partial^2\phi}{\partial x\partial t} \Rightarrow \frac{\partial M}{\partial x} = -\frac{\partial N}{\partial t}$. In the specific case above:

$N(x,t) = r(t), M(x,t) = [p(t)x(t) - g(t)]r(t)$

- **Example** (The Bernoulli Equations)

A first order Bernoulli Equation is of the form: $\dot{x}(t) + p(t)x(t) = q(t)x^n(t)$
(for any integer $n > 1$)

This equation is *not* linear, as one of its terms (i.e. on the right hand side) involves a power (greater than 1) of the unknown function. It can be made linear by first dividing both sides by $x^n(t)$ and then using the substitution: $u(t) = x^{(1-n)}(t) = x(t)x^{-n}(t)$:

$$\dot{x}(t) + p(t)x(t) = q(t)x^n(t) \Rightarrow x^{-n}\dot{x} + px^{1-n} = q$$

(dividing through by $x^n(t)$). Using substitution $u(t) = x^{(1-n)}(t) = x(t)x^{-n}(t)$, note:

$$u(t) = x^{1-n}(t) \Rightarrow \dot{u}(t) = (1-n)x^{-n}(t)\dot{x}(t)$$

Hence the equation becomes:

$$\dot{x}(t) + p(t)x(t) = q(t)x^n(t) \Rightarrow x^{-n}\dot{x} + px^{1-n} = q \Rightarrow \frac{1}{1-n}\dot{u}(t) + p(t)u(t) = q(t)$$

(which *is* linear)

Consider the Bernoulli Eqn: $\dot{x}(t) + \frac{3}{t}x(t) = t^2x^2(t)$.

Dividing through by $x^2(t)$: $x^{-2}\dot{x}(t) + \frac{3}{t}x^{-1}(t) = t^2$

Let: $u(t) = x^{(1-2)}(t) = x^{-1}(t)$

Hence: $u'(t) = -x^{-2}(t)\dot{x}(t)$

So the equation becomes the linear ODE: $x^{-2}\dot{x}(t) + \frac{3}{t}x^{-1}(t) = t^2 \Rightarrow u' - \frac{3}{t}u = -t^2$

So: $r(t) = e^{\int p(t)dt} = e^{-\int \frac{3}{t}dt} = e^{-3\ln t} = e^{\ln t^{-3}} = t^{-3}$

Hence, multiplying both sides by r :

$$t^{-3}u' - \frac{3}{t}t^{-3}u = -t^{-1} \Rightarrow t^{-3}u' + -3t^{-4}u = -t^{-1} \Rightarrow \frac{d}{dt}(t^{-3}u) = -t^{-1}$$

$$\Rightarrow t^{-3}u(t) = -\int \frac{dt}{t} = -\ln t + c \Rightarrow u(t) = t^{-3}(c - \ln t)$$

- **Second-Order linear ODEs with Constant Coefficients**

... Can be solved via the method of undetermined coefficients, whereby one directly inserts a constructed particular (steady-state) solution $x_{ss}(t)$ such that:

If $g(t)$ is:	Let $x_{ss}(t)$ be of the form:
A power-form t^n	$A_0 + A_1t + \dots + A_nt^n$
An exponential form e^{kt}	Ae^{kt}
A sinusoidal term: $\sin\omega t$ (or $\cos\omega t$ or some linear combination of both)	$A \sin \alpha t + B \cos \alpha t$

Whereby one inserts $x_{ss}(t)$ directly into the ODE and solves for the unknown coefficients. In addition, for the homogeneous (transient) term, similar to the integrating factor, one assumes a solution has an exponential form, i.e.: $x(t) = e^{rt}$. Then:

$$\alpha_2 \ddot{x}(t) + \alpha_1 \dot{x}(t) + \alpha_0 x(t) = 0 \Rightarrow (\alpha_2 r^2 + \alpha_1 r + \alpha_0) e^{rt} = 0$$

(since $\ddot{x}(t) = \frac{d^2}{dt^2} e^{rt} = r^2 e^{rt}$, etc.) The term in the parenthesis is the *auxillary equation* of the associated homogenous differential equation. It's of course a quadratic equation. Three cases present themselves:

If discriminant $\alpha_1^2 - 4\alpha_2\alpha_0$ is:	Let $x_r(t)$ be of the form:
> 0 (i.e. two real roots r_1, r_2 exist, where: $r_{1,2} = -\frac{\alpha_1}{2\alpha_2} \pm \frac{1}{2\alpha_2} \sqrt{\alpha_1^2 - 4\alpha_2\alpha_0} \equiv a \pm b$)	$c_1 e^{r_1 t} + c_2 e^{r_2 t} = e^{at} (A \sinh bt + B \cosh bt)$
$= 0$ (i.e. only one root, where: $r = -\frac{\alpha_1}{2\alpha_2}$)	$c_1 e^{rt} + c_2 t e^{rt} = e^{rt} (c_1 + c_2 t)$
< 0 (i.e. two complex roots r_1, r_2 exist, where: $r_{1,2} = -\frac{\alpha_1}{2\alpha_2} \pm \frac{i}{2\alpha_2} \sqrt{\alpha_1^2 - 4\alpha_2\alpha_0} \equiv a \pm ib$)	$c_1 e^{r_1 t} + c_2 e^{r_2 t} = e^{at} (c_1 e^{ibt} + c_2 e^{-ibt})$ $= e^{at} (A \sin bt + B \cos bt) = \alpha e^{at} \sin(bt + \phi)$

- Example (Problem 4, p 80 Sheng)

$$\ddot{x}(t) + 16x = 5 \sin t, \quad x(0) = 0, \quad \dot{x}(0) = 0$$

a.) Find the transient solution $x_r(t)$ (the solution to the homogeneous eqn.)

$$\ddot{x}(t) + 16x(t) = 0 \Rightarrow r^2 + 16 = 0 \Rightarrow r_{1,2} = \pm 4i$$

$$\therefore x_r(t) = c_1 e^{4it} + c_2 e^{-4it} = A \cos 4t + B \sin 4t$$

b.) Use the method of undetermined coefficients to find the steady state solution:

$$\begin{aligned}
x_{ss}(t) &= C \sin t + D \cos t \Rightarrow \ddot{x}(t) = -x_{ss}(t) = -C \sin t - D \cos t \\
\Rightarrow \ddot{x}_{ss}(t) + 16x_{ss}(t) &= -x_{ss}(t) + 16x_{ss}(t) = 15x_{ss}(t) = 15C \sin t + 15D \cos t = 5 \sin t \\
\Rightarrow C &= \frac{5}{15} = \frac{1}{3}, D = 0
\end{aligned}$$

$$c.) x(t) = x_{tr}(t) + x_{ss}(t) = A \cos 4t + B \sin 4t + \frac{1}{3} \sin t$$

Using initial conditions:

$$\begin{aligned}
x(0) &= A = 0 \\
x'(t) &= -4A \sin 4t + 4B \cos 4t + \frac{1}{3} \cos t = 4B \cos 4t + \frac{1}{3} \cos t \\
\Rightarrow x'(0) &= 0 = 4B + \frac{1}{3} \Rightarrow B = -\frac{1}{12}
\end{aligned}$$

$$\text{Hence } x(t) = x_{tr}(t) + x_{ss}(t) = -\frac{1}{12} \sin 4t + \frac{1}{3} \sin t$$

- **Alternative Method: The Variation of Parameters**

A more general technique can be applied in lieu of the method of undetermined coefficients. The method of the variation of parameters (VP) is well suited for cases of order > 2 (this gets cumbersome in the case of the method of undetermined coefficients.) Additionally, the variation of parameters method can take on more complicated forms of $g(t)$.

a.) Solve for the homogeneous (transient) case. We saw earlier that it was:

$$x_{tr}(t) = (A \cos 4t + B \sin 4t)$$

b.) Stipulate a general solution of the form:

$$x(t) = u(t) \cos 4t + v(t) \sin 4t$$

...where the general functions u and v take the place of the constants in the homogeneous case.

c.) Solve the system:

$$\begin{aligned}
u'(t) \cos 4t + v'(t) \sin 4t &= 0 \\
-4u'(t) \sin 4t + 4v'(t) \cos 4t &= 5 \sin t
\end{aligned}$$

(the bottom equation includes the derivatives of the t -dependent factors in the homogeneous solution, i.e. the derivatives of $\cos 4t, \sin 4t$)

Expressing this in matrix notation:

$$\begin{bmatrix} \cos 4t & \sin 4t \\ -4 \sin 4t & 4 \cos 4t \end{bmatrix} \begin{bmatrix} u'(t) \\ v'(t) \end{bmatrix} = \begin{bmatrix} 0 \\ 5 \sin t \end{bmatrix}$$

The determinant of this matrix is called the *Wronskian*. Evaluating:

$$4 \cos^2 t - (-4 \sin^2 t) = 4(\cos^2 t + \sin^2 t) = 4$$

Hence according to Cramer's Rule:

$$\begin{aligned} u'(t) &= \frac{1}{4} \begin{vmatrix} 0 & \sin 4t \\ 5 \sin t & 4 \cos 4t \end{vmatrix} = -\frac{1}{4} 5 \sin t \sin 4t = -\frac{5}{4} \sin t \sin 4t = -\frac{5}{8} [\cos(t-4t) - \cos(t+4t)] \\ &= -\frac{5}{8} (\cos(-3t) - \cos 5t) = -\frac{5}{8} (\cos 3t - \cos 5t) = \frac{5}{8} (\cos 5t - \cos 3t) \end{aligned}$$

$$\begin{aligned} v'(t) &= \frac{1}{4} \begin{vmatrix} \cos 4t & 0 \\ -4 \sin t & 5 \sin t \end{vmatrix} = \frac{1}{4} 5 \cos 4t \sin t = \frac{5}{4} \sin t \cos 4t = \frac{5}{8} [\sin(t+4t) + \sin(t-4t)] \\ &= \frac{5}{8} (\sin(5t) + \sin(-3t)) = \frac{5}{8} (\sin 5t - \sin 3t) \end{aligned}$$

d.) Integrating:

$$\begin{aligned} u(t) &= \int \frac{5}{8} (\cos 5t - \cos 3t) dt = \frac{1}{8} \sin 5t - \frac{5}{24} \sin 3t + c_1 \\ v(t) &= \int \frac{5}{8} (\sin 5t - \sin 3t) dt = -\frac{1}{8} \cos 5t + \frac{5}{24} \cos 3t + c_2 \end{aligned}$$

e.) So the general solution is:

$$x(t) = \left(-\frac{5}{24} \sin 3t + \frac{1}{8} \sin 5t + c_1\right) \cos 4t + \left(-\frac{1}{8} \cos 5t + \frac{5}{24} \cos 3t + c_2\right) \sin 4t$$

f.) Finding the coefficients using initial conditions:

$$\begin{aligned} x(0) &= \left(-\frac{5}{24} \sin 0 + \frac{1}{8} \sin 0 + c_1\right) \cos 0 + \left(-\frac{1}{8} \cos 0 + \frac{5}{24} \cos 0 + c_2\right) \sin 0 = 0 \Rightarrow c_1 = 0 \\ x'(t) &= \left(-\frac{5}{8} \cos 3t + \frac{5}{8} \cos 5t\right) \cos 4t - 4\left(-\frac{5}{24} \sin 3t + \frac{1}{8} \sin 5t\right) \sin 4t + \\ &\quad \left(\frac{5}{8} \sin 5t - \frac{5}{8} \sin 3t\right) \sin 4t + 4\left(-\frac{1}{8} \cos 5t + \frac{5}{24} \cos 3t + c_2\right) \cos 4t \\ \Rightarrow x'(0) &= \left(-\frac{5}{8} \cos 0 + \frac{5}{8} \cos 0\right) \cos 0 - 4\left(-\frac{5}{24} \sin 0 + \frac{1}{8} \sin 0\right) \sin 0 + \\ &\quad \left(\frac{5}{8} \sin 0 - \frac{5}{8} \sin 0\right) \sin 0 + 4\left(-\frac{1}{8} \cos 0 + \frac{5}{24} \cos 0 + c_2\right) \cos 0 = 0 \\ \Rightarrow 0 &= 4\left(-\frac{1}{8} + \frac{5}{24} + c_2\right) \Rightarrow c_2 = -\frac{1}{12} \end{aligned}$$

So:

$$\begin{aligned}
x(t) &= \left(-\frac{5}{24} \sin 3t + \frac{1}{8} \sin 5t\right) \cos 4t + \left(-\frac{1}{8} \cos 5t + \frac{5}{24} \cos 3t - \frac{1}{12}\right) \sin 4t \\
&= \frac{1}{24} \{-5 \sin 3t \cos 4t + 3 \sin 5t \cos 4t - 3 \cos 5t \sin 4t + 5 \cos 3t \sin 4t - 2 \sin 4t\} \\
&= \frac{1}{24} (-5 \sin 3t \cos 4t + 3 \sin 5t \cos 4t - 3 \cos 5t \sin 4t + 5 \cos 3t \sin 4t) - \frac{1}{12} \sin 4t \\
&= \frac{1}{24} [5(\cos 3t \sin 4t - \sin 3t \cos 4t) - 3(\cos 5t \sin 4t - \sin 5t \cos 4t)] - \frac{1}{12} \sin 4t \\
&= \frac{1}{24} [5 \sin(4t - 3t) - 3 \sin(4t - 5t)] - \frac{1}{12} \sin 4t \\
&= \frac{1}{24} [5 \sin t - 3 \sin(-t)] - \frac{1}{12} \sin 4t \\
&= \frac{1}{24} [5 \sin t + 3 \sin t] - \frac{1}{12} \sin 4t \\
&= \frac{1}{24} [8 \sin t] - \frac{1}{12} \sin 4t \\
&= \frac{1}{3} \sin t - \frac{1}{12} \sin 4t
\end{aligned}$$

- Example (Problem 6, p 80 Sheng)

$$\ddot{x}(t) + 4\dot{x}(t) + 8x(t) = 4 \cos t + 7 \sin t, x(0) = 1, \dot{x}(0) = -1$$

a.) Find the transient solution by solving the homogeneous eqn.:

$$\begin{aligned}
\ddot{x}(t) + 4\dot{x}(t) + 8x(t) = 0 &\Rightarrow r^2 + 4r + 8 = 0 \Rightarrow r_{1,2} = -2 \pm 2i = -2(1 \mp i) \\
x_{tr}(t) &= e^{-2t} (c_1 e^{2it} + c_2 e^{-2it}) = e^{-2t} (A \sin 2t + B \cos 2t)
\end{aligned}$$

b.) Find the particular (steady-state) solution. Since g is a linear combination of sinusoidal terms, let: $x_{ss} = C \sin t + D \cos t$

Inserting into right hand side:

$$\begin{aligned}
\ddot{x}_{ss}(t) + 4\dot{x}_{ss}(t) + 8x_{ss}(t) &= -(C \sin t + D \cos t) + 4(C \cos t - D \sin t) + 8(C \sin t + D \cos t) \\
&= 4 \cos t + 7 \sin t, \\
\Rightarrow -C - 4D + 8C &= 7 \Rightarrow 7C - 4D = 7 \Rightarrow 49C - 28D = 49 \\
\Rightarrow -D + 4C + 8D &= 4 \Rightarrow 4C + 7D = 4 \Rightarrow 16C + 28D = 16 \Rightarrow C = 1, D = 0
\end{aligned}$$

(equating coefficients of sine and cosine terms, and solving the 2x2 system respectively)

$$\text{Hence: } x_{ss} = \sin t$$

So:

$$\begin{aligned}
x(t) &= x_{tr}(t) + x_{ss}(t) = e^{-2t} (A \sin 2t + B \cos 2t) + \sin t \\
\dot{x}(t) &= -2e^{-2t} (A \sin 2t + B \cos 2t) + e^{-2t} (2A \cos 2t - 2B \sin 2t) + \cos t \\
&= 2e^{-2t} \{-\sin 2t(A + B) + \cos 2t(A - B)\} + \cos t
\end{aligned}$$

Hence:

$$\begin{aligned}
x(0) &= e^0 (A \sin 0 + B \cos 0) + \sin 0 = B = 1 \\
\dot{x}(0) &= 2e^0 \{-\sin 0(A + B) + \cos 0(A - B)\} + \cos 0 = 2A - 2B + 1 = -1 \Rightarrow A - B = -1 \\
&\Rightarrow A - 1 = -1 \Rightarrow A = 0
\end{aligned}$$

So:

$$x(t) = x_{tr}(t) + x_{ss}(t) = e^{-2t} (A \sin 2t + B \cos 2t) + \sin t = e^{-2t} \cos 2t + \sin t$$

- **Alternate Method (Variation of Parameters)**

In this example, however, due to the nature of the functions involved, the VP method isn't more efficient. I sketch out the strategy here, leaving the algebraic details aside. The primary purpose is to show that it's a systematic (hence programmable) procedure, whereas the method of undetermined coefficients is ad-hoc.

g.) Solve for the homogeneous (transient) case. (We saw earlier that it was:

$$x_{tr}(t) = e^{-2t} (c_1 e^{2it} + c_2 e^{-2it}) = e^{-2t} (A \sin 2t + B \cos 2t)$$

h.) Stipulate a general solution of the form:

$$x(t) = e^{-2t} (u(t)e^{2it} + v(t)e^{-2it}) = u(t)e^{-2(1-i)t} + v(t)e^{-2(1+i)t}$$

where the general functions u and v take the place of the constants in the homogeneous case.

c.) Solve the system:

$$\begin{aligned}
&u'(t)e^{-2t+2it} + v'(t)e^{-2t-2it} = 0 \\
&2(-1+i)u'(t)e^{-2t+2it} - 2(1+i)v'(t)e^{-2t-2it} = 4\cos t + 7\sin t
\end{aligned}$$

(the bottom equation includes the derivatives of the t -dependent factors in the homogeneous solution, i.e. the derivatives of $e^{-2(1 \pm i)t}$)

Expressing this in matrix notation:

$$\begin{aligned}
&\begin{bmatrix} e^{-2t(1-i)} & e^{-2t(1+i)} \\ -2(1-i)e^{-2t(1-i)} & -2(1+i)e^{-2t(1+i)} \end{bmatrix} \begin{bmatrix} u' \\ v' \end{bmatrix} = \begin{bmatrix} 0 \\ 4\cos t + 7\sin t \end{bmatrix} = \begin{bmatrix} 0 \\ 2(e^{it} + e^{-it}) - \frac{7}{2}i(e^{it} - e^{-it}) \end{bmatrix} \\
&= \begin{bmatrix} 0 \\ e^{it}(2 - \frac{7}{2}i) + e^{-it}(2 + \frac{7}{2}i) \end{bmatrix}
\end{aligned}$$

The determinant of this matrix is called the *Wronskian*. Evaluating:

$$W(t) = -2(1+i)e^{-4t} + 2(1-i)e^{-4t} = -4ie^{-4t}$$

Hence according to Cramer's Rule:

$$\begin{aligned} u' &= \frac{\begin{vmatrix} 0 & e^{-2t(1+i)} \\ e^{it}(2-\frac{7}{2}i) + e^{-it}(2+\frac{7}{2}i) & -2(1+i)e^{-2t(1+i)} \end{vmatrix}}{W(t)} = \frac{(2-\frac{7}{2}i)e^{(-2-i)t} + (2+\frac{7}{2}i)e^{(-2-3i)t}}{-4ie^{-4t}} \\ &= \frac{i}{4}e^{4t} \left\{ e^{-2t} \left[(2-\frac{7}{2}i)e^{-it} + (2+\frac{7}{2}i)e^{-3it} \right] \right\} = \frac{1}{4}e^{2t} \left[(\frac{7}{2}+2i)e^{-it} + (-\frac{7}{2}+2i)e^{-3it} \right] \\ &= (\frac{7}{8} + \frac{i}{2})e^{(2-i)t} + (-\frac{7}{8} + \frac{i}{2})e^{(2-3i)t} \end{aligned}$$

$$\begin{aligned} v' &= \frac{\begin{vmatrix} e^{-2t(1-i)} & 0 \\ 2(i-1)e^{-2t(1-i)} & e^{it}(2-\frac{7}{2}i) + e^{-it}(2+\frac{7}{2}i) \end{vmatrix}}{W(t)} = \frac{(2-\frac{7}{2}i)e^{(-2+3i)t} + (2+\frac{7}{2}i)e^{(-2-i)t}}{-4ie^{-4t}} \\ &= \frac{i}{4}e^{4t} \left\{ e^{-2t} \left[(2-\frac{7}{2}i)e^{3it} + (2+\frac{7}{2}i)e^{-it} \right] \right\} = \frac{1}{4}e^{2t} \left[(\frac{7}{2}+2i)e^{3it} + (-\frac{7}{2}+2i)e^{-it} \right] \\ &= (\frac{7}{8} + \frac{i}{2})e^{(2+3i)t} + (-\frac{7}{8} + \frac{i}{2})e^{(2-i)t} \end{aligned}$$

Hence, integrating:

$$\begin{aligned} u(t) &= \int (\frac{7}{8} + \frac{i}{2})e^{(2-i)t} dt + \int (-\frac{7}{8} + \frac{i}{2})e^{(2-3i)t} dt = \frac{\frac{1}{8}(7+4i)}{(2-i)}e^{(2-i)t} + \frac{\frac{1}{8}(-7+4i)}{(2-3i)}e^{(2-3i)t} + c_1 \\ &= \frac{(7+4i)}{8(2-i)} \cdot \frac{(2+i)}{(2+i)}e^{(2-i)t} + \frac{(-7+4i)}{8(2-3i)} \cdot \frac{(2+3i)}{(2+3i)}e^{(2-3i)t} + c_1 = \frac{10+15i}{8(5)}e^{(2-i)t} + \frac{-26+13i}{8(13)}e^{(2-3i)t} + c_1 \\ &= \frac{1}{8} \left[(2+3i)e^{(2-i)t} + (-2+i)e^{(2-3i)t} \right] + c_1 = \frac{e^{2t}}{8} \left[(2+3i)e^{-it} + (-2+i)e^{-3it} \right] + c_1 \end{aligned}$$

$$\begin{aligned} v(t) &= \int (\frac{7}{8} + \frac{i}{2})e^{(2+3i)t} dt + \int (-\frac{7}{8} + \frac{i}{2})e^{(2-i)t} dt = \frac{\frac{1}{8}(7+4i)}{(2+3i)}e^{(2+3i)t} + \frac{\frac{1}{8}(-7+4i)}{(2-i)}e^{(2-i)t} + c_2 \\ &= \frac{(7+4i)}{8(2+3i)} \cdot \frac{(2-3i)}{(2-3i)}e^{(2+3i)t} + \frac{(-7+4i)}{8(2-i)} \cdot \frac{(2+i)}{(2+i)}e^{(2-i)t} + c_2 = \frac{26-13i}{8(13)}e^{(2+3i)t} + \frac{-18+i}{8(5)}e^{(2-i)t} + c_2 \\ &= \frac{1}{8} \left[(2-i)e^{(2+3i)t} + (-\frac{18}{5} + \frac{1}{5}i)e^{(2-i)t} \right] + c_2 = \frac{e^{2t}}{8} \left[(2-i)e^{3it} + \frac{1}{5}(-18+i)e^{-it} \right] + c_2 \end{aligned}$$

$$\text{So: } u(t)e^{-2(1-i)t} = \frac{1}{8} \left[(2+3i)e^{it} + (-2+i)e^{-it} \right] + c_1e^{-2(1-i)t}$$

$$v(t)e^{-2(1-i)t} = \frac{1}{8} \left[(2+3i)e^{it} + (-2+i)e^{-it} \right] + c_1e^{-2(1-i)t}$$