MA360 **Handout 6a (week of Feb 20)**
Sp2006 Kallfelz

Assignment: page 80 Sheng (Exercise 7) Problems 1,3,5,7

- Example (#2, Sheng p 68)

$$F(s)=\frac{11s^2-2s+5}{(s-2)(2s-1)(s+1)}=\frac{A}{(s-2)}+\frac{B}{(2s-1)}+\frac{C}{(s+1)}$$

Using Heaviside Cover Method:

$$s=2\Rightarrow A=\frac{11\cdot 2^2-2\cdot 2+5}{(2\cdot 2-1)(2+1)}=\frac{45}{9}=5$$

$$s=\tfrac{1}{2}\Rightarrow B=\frac{11\cdot\left(\frac{1}{2}\right)^2-2\cdot\left(\frac{1}{2}\right)+5}{\left(\frac{1}{2}-2\right)\left(\frac{1}{2}+1\right)}=\frac{27}{-9}=-3$$

$$s=-1\Rightarrow C=\frac{11\cdot 1-2\cdot(-1)+5}{(-1-2)(2\cdot(-1)-1)}=\frac{18}{9}=2$$

So:

$$F(s)=\frac{11s^2-2s+5}{(s-2)(2s-1)(s+1)}=\frac{5}{(s-2)}+\frac{-3}{2\left(s-\frac{1}{2}\right)}+\frac{2}{(s+1)}=5L\left\{e^{2t}\right\}-3L\left\{e^{\frac{1}{2}t}\right\}+2L\left\{e^{-t}\right\}$$

$$=L\left\{5e^{2t}-\tfrac{3}{2}e^{\frac{t}{2}}+2e^{-t}\right\}\Rightarrow f(t)=5e^{2t}-\tfrac{3}{2}e^{\frac{t}{2}}+2e^{-t}$$

- Example (#5, Sheng p 68)

$$F(s)=\frac{(s+1)e^{-4s}}{\left(6s^2+7s+2\right)}=L\{u(t-4)f(t-4)\}$$ (according to THM8)

where: $L\{f(t)\}=\frac{(s+1)}{\left(6s^2+7s+2\right)}=\frac{(s+1)}{6\left(s+\frac{1}{2}\right)\left(s+\frac{2}{3}\right)}=\frac{(s+1)/6}{\left(s+\frac{1}{2}\right)\left(s+\frac{2}{3}\right)}=\frac{A}{\left(s+\frac{1}{2}\right)}+\frac{B}{\left(s+\frac{2}{3}\right)}$

Using Heaviside Cover Method:

$$s=-\tfrac{1}{2}\Rightarrow A=\frac{\left(-\frac{1}{2}+1\right)/6}{\left(-\frac{1}{2}+\frac{2}{3}\right)}=\frac{1}{2}$$

$$s=-\tfrac{2}{3}\Rightarrow B=\frac{\left(-\frac{2}{3}+1\right)/6}{\left(-\frac{2}{3}+\frac{1}{2}\right)}=-\frac{1}{3}$$

So:

$$L\{f(t)\}=\frac{1}{2}\cdot\frac{1}{\left(s+\frac{1}{2}\right)}-\frac{1}{3}\cdot\frac{1}{\left(s+\frac{2}{3}\right)}=\frac{1}{2}L\left\{e^{-\frac{t}{2}}\right\}-\frac{1}{3}L\left\{e^{-\frac{2}{3}t}\right\}=L\left\{\frac{1}{2}\cdot e^{-\frac{t}{2}}-\frac{1}{3}\cdot e^{-\frac{2}{3}t}\right\}$$

Hence:

$$F(s)=\frac{(s+1)e^{-4s}}{\left(6s^2+7s+2\right)}=L\{u(t-4)f(t-4)\}=L\left\{u(t-4)\left[\frac{1}{2}e^{-\frac{1}{2}(t-4)}-\frac{1}{3}e^{-\frac{2}{3}(t-4)}\right]\right\}$$

So: $L^{-1}\{F(s)\}=\frac{1}{2}e^{-\frac{1}{2}(t-4)}-\frac{1}{3}e^{-\frac{2}{3}(2-4)}$, for $t>4$ (0 otherwise)

- Example (#11, Sheng p 68)

$$F(s)=\frac{s}{(s-1)^3}=\frac{A}{(s-1)}+\frac{B}{(s-1)^2}+\frac{C}{(s-1)^3}\Rightarrow s=A(s-1)^2+B(s-1)+C$$
$$s=1\Rightarrow 1=C$$
$$s=0\Rightarrow 0=A-B+1\Rightarrow 1=B-A$$
$$s=2\Rightarrow 2=A+B+1\Rightarrow 1=B+A$$
$$\Rightarrow B=1, A=0$$

So: $F(s)=\frac{1}{(s-1)^2}+\frac{1}{(s-1)^3}=F_1(s-1)+F_2(s-1)=L\{e^t f_1(t)\}+L\{e^t f_2(t)\}$

where: (by THM7): $L\{f_1(t)\}=\frac{1}{s^2}=L\{t\}$
$L\{f_2(t)\}=\frac{1}{s^3}=\frac{1}{2}\cdot\frac{2}{t^3}=\frac{1}{2}\cdot L\{t^2\}$

So: $F(s)=L\{e^t f_1(t)\}+L\{e^t f_2(t)\}=L\{e^t(t+\frac{1}{2}t^2)\}\Rightarrow f(t)=L^{-1}\{F(s)\}=e^t(t+\frac{1}{2}t^2)$

- Example (#14, Sheng p 68)

$$F(s)=\frac{s^2+1}{s(s+1)^2}=\frac{A}{(s-0)}+\frac{B}{(s+1)}+\frac{C}{(s+1)^2}\Rightarrow s^2+1=A(s+1)^2+Bs(s+1)+Cs$$
$$s=-1\Rightarrow 2=-C\Rightarrow C=-2$$
$$s=0\Rightarrow 1=A$$
$$s^1: 0=2A+2B+C\Rightarrow 0=2+2B-2\Rightarrow 0=B$$
$$check: s^2: 1=A+B\Rightarrow 1=1+B\Rightarrow B=0$$

So: $F(s)=\frac{1}{s}-2\cdot\frac{1}{(s+1)^2}=L\{1\}-2G(s+1)$, where:
$G(s+1)=\frac{1}{(s+1)^2}=L\{e^{-t}g(t)\}\Rightarrow L\{g(t)\}=\frac{1}{s^2}=L\{t\}$ (according to THM7)

So:
$F(s)=L\{1\}-2G(s+1)=L\{1\}-2L\{e^{-t}t\}=L\{1-2e^{-t}t\}\Rightarrow f(t)=L^{-1}\{F(s)\}=1-2te^{-t}$

- Example (#17, Sheng p 68)

$$F(s)=\frac{250}{s^3(s^2-4s+5)}=\frac{A}{s}+\frac{B}{s^2}+\frac{C}{s^3}+\frac{D_1s+D_2}{(s^2-4s+5)}\Rightarrow$$
$$250=As^2(s^2-4s+5)+Bs(s^2-4s+5)+C(s^2-4s+5)+(D_1s+D_2)s^3$$
$$s=0\Rightarrow 250=5C\Rightarrow C=50$$

Recall, hints from Handout 4a (which are shown in detail in the Key for Assignment I):

1. Complete the square on the irreducible in (Eqn A), and then set $s = 2$. You'll end up with a formula: $B = 100 - 2C - 8D_1 - 4D_2$ (Eqn B)
2. The roots of the irreducible are: $s_{1,2} = 2 \pm i$. So setting $s = s_1$ (for instance) in (Eqn A) gives us an equation just in terms of B_1 and B_2. (I.e., you'll end up with an equation: $250 = -7D_1 + 2D_1 + i(24D_1 + 11D_2)$ Equating real terms on the right hand side with the left hand side, and imaginary terms on the right hand side[1] with those on on the left hand side gives two equations and two unknowns. Solving: $D_1 = -22, D_2 = 48$
3. Differentiate both sides of Eqn (A) by s, and then set $s = 0$. You'll get: $B = 40$.
4. Now solving for A in (Eqn B): $A = 22$

Now, instead of finding the inverse LT for the quadratic irreducible term using first principles, suppose instead we use the formula in Lemma1:

$$L^{-1}\left\{\frac{s+a}{(s+b)^2+\omega^2}\right\} = \frac{k}{\omega}e^{-bt}\sin(\omega t+\phi)\text{, where: } k = \sqrt{(a-b)^2+\omega^2}, \phi = \arctan\left(\frac{\omega}{a-b}\right)$$

$$F(s) = \frac{250}{s^3(s^2-4s+5)} = \frac{22}{s} + \frac{40}{s^2} + \frac{50}{s^3} + \frac{-22s+48}{(s^2-4s+5)} \Rightarrow 22L\{1\} + 40L\{t\} + \frac{50}{2}L\{t^2\} + G(s)$$
$$= L\{22 + 40t + 25t^2\} + G(s)$$

where: $G(s) = \frac{-22\left(s-\frac{24}{11}\right)}{(s^2-4s+5)} = -22\left\{\frac{s-\frac{24}{11}}{(s-2)^2+1^2}\right\}$

so: $a = -\frac{24}{11}, b = -2, \omega = 1 \Rightarrow k = \sqrt{\left(-\frac{24}{11}+2\right)^2+1} = \sqrt{\frac{4}{121}+\frac{121}{121}} = \sqrt{\frac{125}{121}} = \frac{5}{11}\sqrt{5}$

So:

$$L^{-1}\left\{\frac{s+a}{(s+b)^2+\omega^2}\right\} = \frac{k}{\omega}e^{-bt}\sin(\omega t+\phi) = \frac{k}{\omega}e^{-bt}(\sin\omega t\cos\phi + \cos\omega t\sin\phi)$$
$$= \frac{5}{11}\sqrt{5}e^{2t}\left[\sin t\cos\left(\tan^{-1}\left(-\frac{1}{\frac{2}{11}}\right)\right) + \cos t\sin\left(\tan^{-1}\left(\frac{1}{-\frac{2}{11}}\right)\right)\right]$$
$$= \frac{5}{11}\sqrt{5}e^{2t}\left[\sin t\cos\left(\tan^{-1}\left(-\frac{11}{2}\right)\right) + \cos t\sin\left(\tan^{-1}\left(-\frac{11}{2}\right)\right)\right]$$
$$= \frac{5}{11}\sqrt{5}e^{2t}\left[\sin t\left(\pm\frac{2}{5\sqrt{5}}\right) + \cos t\left(\mp\frac{11}{5\sqrt{5}}\right)\right] = \frac{1}{11}e^{2t}(\pm 2\sin t \mp 11\cos t)$$

Note: the sign ambiguity results depends on whether the phase angle is acute (i.e. $|\phi| < \frac{\pi}{2}$) or obtuse (i.e. $\frac{\pi}{2} < \phi < \pi$)

$$F(s) = L\{22 + 40t + 25t^2\} - 22L\left\{\frac{1}{11}e^{2t}(\pm 2\sin t \mp 11\cos t)\right\}$$

So: $\Rightarrow L^{-1}\{F(s)\} = 22 + 40t + 25t^2 + 2e^{2t}(\mp 2\sin t \pm 11\cos t)$

$$= 22 + 40t + 25t^2 + 2e^{2t}(2\sin t - 11\cos t)$$

(Select the obtuse case)

[1] On the right hand side, the imaginary terms =0, obviously