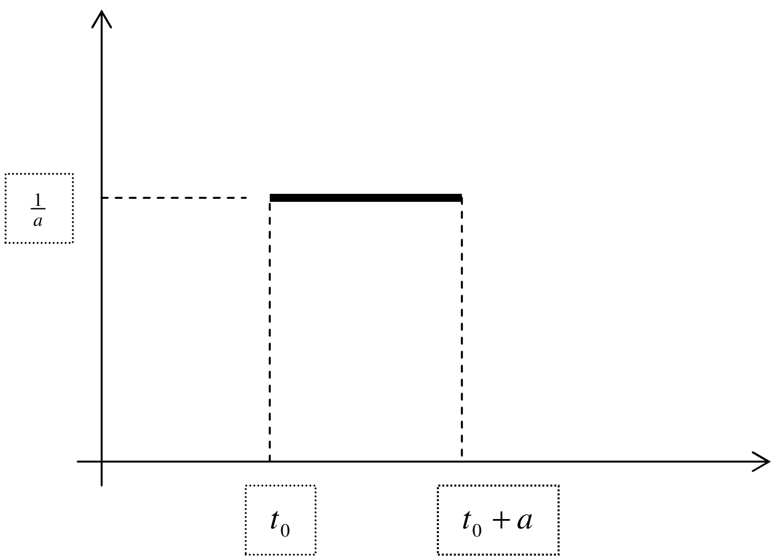
MA360 **Handout 4b (week of Jan 30)**
Sp2006 Kallfelz

The Unit Impulse Function[1]

Consider square wave of width *a* and height $^1/_a$, anchored at t_0, i.e.:

$$\Delta_a(t_0) = \tfrac{1}{a}\left(u(t-t_0) - u(t-(t_0+a))\right) = \frac{u(t-t_0) - u(t-t_0-a)}{a}$$



Then by definition: $\delta(x-x_0) = \lim_{a\to 0} \Delta_a(x_0) = \lim_{a\to 0} \frac{(u(x-x_0) - u(x-x_0-a))}{a}$

Lemma 1: $\frac{d}{dt}u(t-t_0) = \delta(t-t_0)$

Proof: $\frac{d}{dt}u(t-t_0) = \lim_{h\to 0} \frac{u(t-t_0+h) - u(t-t_0)}{h}$

Define: $h = -a$. Then:

$$\frac{d}{dt}u(t-t_0) = \lim_{a\to 0} \frac{u(t-t_0-a) - u(t-t_0)}{-a} = \lim_{a\to 0} \frac{u(t-t_0) - u(t-t_0-a)}{a} = \delta(t-t_0)$$

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[1] Also known as the 'Dirac delta-function,' in homage to physicist Paul Dirac.

Lemma 2: $L\{\delta(t-t_0)\} = e^{-st_0}$

Proof:

$$L\{\delta(t-t_0)\} = L\left\{\lim_{a\to 0}\frac{u(t-t_0)-u(t-t_0-a)}{a}\right\} = \lim_{a\to 0}\tfrac{1}{a}\{L\{u(t-t_0)\}-L\{u(t-t_0-a)\}\}$$

$$= \lim_{a\to 0}\tfrac{1}{a}\left\{\frac{e^{-t_0 s}}{s}-L\{u(t-(t_0+a))\}\right\} = \lim_{a\to 0}\tfrac{1}{a}\left\{\frac{e^{-t_0 s}}{s}-\frac{e^{-s(t_0+a)}}{s}\right\} = e^{-t_0 s}\lim_{a\to 0}\frac{(1-e^{-as})}{as}$$

$$= e^{-t_0 s}\lim_{a\to 0}\frac{\frac{d}{da}(1-e^{-as})}{\frac{d}{da}(as)} = e^{-t_0 s}\lim_{a\to 0}\frac{se^{-as}}{s} = e^{-t_0 s}\lim_{a\to 0}e^{-as} = e^{-t_0 s}$$

(Note how L'Hopital's Rule must be invoked to evaluate this above limit)

So the LT for the unit impulse function is a decaying exponential. Hence the inverse LT of a decaying exponential is the unit impulse function. Note: We can also verify the result in Lemma2 using THM4:

Since: $\frac{d}{dt}u(t-t_0) = \delta(t-t_0)$ then: $u(t-t_0) = \int_{t_0}^{t}\delta(\omega-t_0)d\omega$

Hence according to Thm4: $L\left\{\int_{t_0}^{t}\delta(\omega-t_o)d\omega\right\} = \frac{1}{s}L\{\delta(t-t_0)\}$

But we also know that: $L\left\{\int_{t_0}^{t}\delta(\omega-t_o)d\omega\right\} = L\{u(t-t_0)\} = \frac{e^{-t_0 s}}{s}$

Hence: $L\left\{\int_{t_0}^{t}\delta(\omega-t_o)d\omega\right\} = L\{u(t-t_0)\} = \frac{e^{-t_0 s}}{s} = \frac{1}{s}L\{\delta(t-t_0)\} \Rightarrow L\{\delta(t-t_0)\} = e^{-t_0 s}$

In particular, if $t_0 = 0$, then: $L\{\delta(t-0)\} = L\{\delta(t)\} = e^{-s0} = 1$

Or: $L^{-1}\{k\} = kL^{-1}\{1\} = k\delta(t)$

(I.e., the inverse LT of any constant function *k* (a horizontal line in *s*-space) is the unit impulse function with amplitude *k*, i.e. an infinite vertical ray in *t*-space!)

<u>Example:</u> Use the above result to find[2]: $L^{-1}\left[\frac{s^2}{s^2-a^2}\right]$

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[2] This was on part of the original formulation of Problem IIb, Assignment I. But I decided to make it simpler and remove the squared term.

Solution: According to Thm2:

$$F(s)=\frac{s^2}{s^2-a^2}=s\frac{s}{s^2-a^2}=sL\{\cosh at\}\Rightarrow sL\{\cosh at\}=L\{\tfrac{d}{dt}\cosh at\}+\cosh(a\cdot 0)$$

Hence: $$F(s)=\frac{s^2}{s^2-a^2}=sL\{\cosh at\}=L\{a\sinh at\}+1$$

So: $$L^{-1}\left[\frac{s^2}{s^2-a^2}\right]=L^{-1}\{L\{a\sinh at\}+1\}=a\sinh at+L^{-1}\{1\}=a\sinh at+\delta(t)$$

Lemma 3: $$\int_{-\infty}^{\infty} f(t)\delta(t-t_0)=f(t_0)$$

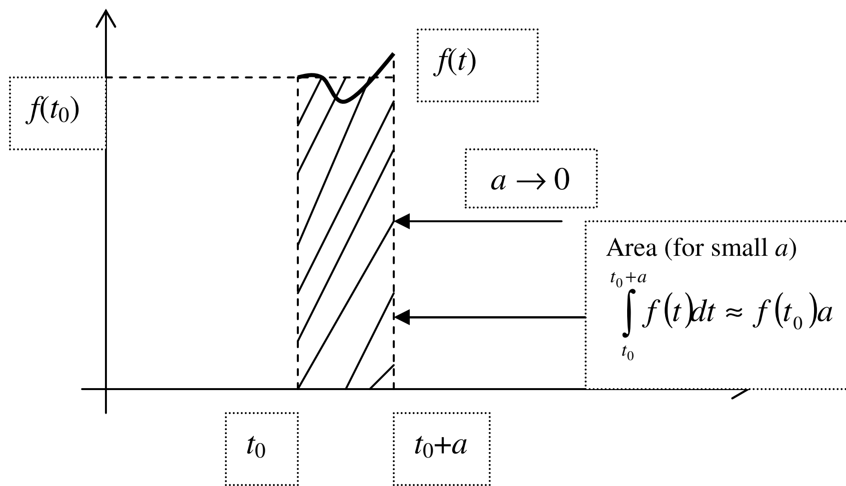
Proof:

$$\int_{-\infty}^{\infty} f(t)\delta(t-t_0)=\int_{-\infty}^{\infty} f(t)\lim_{a\to 0}\left[\frac{u(t-t_0)-u(t-t_0-a)}{a}\right]$$

$$=\lim_{a\to 0}\tfrac{1}{a}\left\{\int_{-\infty}^{\infty} f(t)u(t-t_0)dt-\int_{-\infty}^{\infty} f(t)u(t-t_0-a)dt\right\}=\lim_{a\to 0}\tfrac{1}{a}\left\{\int_{t_0}^{\infty} f(t)dt-\int_{t_0+a}^{\infty} f(t)dt\right\}$$

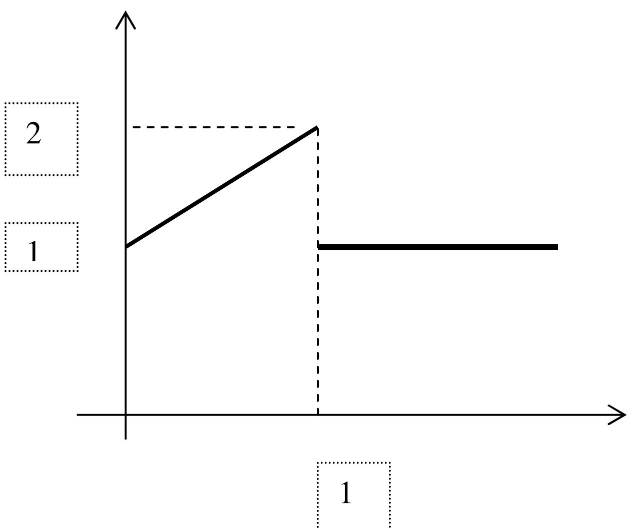
$$=\lim_{a\to 0}\tfrac{1}{a}\int_{t_0}^{t_0+a} f(t)dt=\lim_{a\to 0}\tfrac{1}{a}[f(t_0)a]=f(t_0)$$

The last step in the proof involved approximating the integral $\int_{t_0}^{t_0+a} f(t)dt$ with the area of a rectangle of height $f(t_0)$ and width a. Note that in the limit $a\to 0$, the approximation becomes exact. (See figure next page) :



- **<u>Example (Modified 2d) Shen, p53)</u>**

Find $f(t), f^{/}(t)$ and the associated LTs of $f(t), f^{/}(t)$:



a.)

$f(t) = (t+1)[u(t) - u(t-1)] + 1 \cdot u(t-1) = tu(t) + u(t) - tu(t-1) - u(t-1) + u(t-1)$

$= (t+1)u(t) - tu(t-1)$

b.)

$$f'(t) = (t+1)\tfrac{d}{dt}u(t) + u(t) - t\tfrac{d}{dt}u(t-1) - u(t-1)$$
$$= (t+1)\delta(t) + u(t) - t\delta(t-1) - u(t-1) = t\delta(t) + \delta(t) + u(t) - t\delta(t-1) - u(t-1)$$
$$= \delta(t) - t\delta(t-1) + u(t) - u(t-1)$$

(Recall remark, p51, Sheng: $t\delta(t) = 0$)

c.)

$$L\{f(t)\} = L\{(t+1)u(t) - tu(t-1)\} = L\{tu(t)\} + L\{u(t)\} - L\{tu(t-1)\}$$
$$= (-1)\tfrac{d}{ds}L\{u(t)\} + L\{u(t)\} - (-1)\tfrac{d}{ds}L\{u(t-1)\}$$
$$= -\tfrac{d}{ds}\left(\tfrac{1}{s}\right) + \tfrac{1}{s} + \tfrac{d}{ds}\left(\tfrac{e^{-s}}{s}\right) = \tfrac{1}{s^2} + \tfrac{1}{s} + \tfrac{-se^{-s} - e^{-s}}{s^2} = \tfrac{1}{s^2}\left(1 + s - se^{-s} - e^{-s}\right)$$

d.)

$$L\{f'(t)\} = L\{\delta(t) - t\delta(t-1) + u(t) - u(t-1)\} = L\{\delta(t)\} - L\{t\delta(t-1)\} + L\{u(t)\} - L\{u(t-1)\}$$
$$= 1 - (-1)\tfrac{d}{ds}L\{\delta(t-1)\} + L\{u(t)\} - L\{u(t-1)\} = 1 + \tfrac{d}{ds}\left(e^{-s}\right) + \tfrac{1}{s} - \tfrac{e^{-s}}{s} = 1 - e^{-s} + \tfrac{1}{s}\left(1 - e^{-s}\right)$$
$$= \tfrac{1}{s}\left(s - se^{-s} + 1 - e^{-s}\right) = \tfrac{1}{s}\left(1 + s - se^{-s} - e^{-s}\right)$$

Note the obvious connection between the answer in d) and c). This can be substantiated using THM2:

$$L\{f'(t)\} = sL\{f(t)\} - f(0)$$

Here: $f(0) = (0+1)u(0) - 0u(0-1) = 0$ (since $u(t) = 0$ for all $t \leq 0$)

And as results c) and d) clearly show:

$$L\{f'(t)\} = sL\{f(t)\}$$

Since: $L\{f'(t)\} = \tfrac{1}{s}\left(1 + s - se^{-s} - e^{-s}\right)$

$L\{f(t)\} = \tfrac{1}{s^2}\left(1 + s - se^{-s} - e^{-s}\right)$