MA360 **Handout 3b (week of Jan 23)**
Sp2006 Kallfelz

Assignment: P 41 Sheng: 1,2,4,5(a), b), c)),6(b), c)), 7a)

Second session of Class: Remaining Material in Ch I, Sheng

Lemma 3b.1: $\lim_{s\to\infty} L[f(t)] = \lim_{s\to\infty} F(s) = 0$, for any continuous[1] f.

Proof: $$\left|L[f(t)]\right| = \left|\int_0^\infty f(t)e^{-st}\,dt\right| \le \int_0^\infty \left|f(t)\right|e^{-st}\,dt$$

Moreover, for any continuous $f(t)$, there exist M, α, t_0 such that:

$$\left|f(t)\right| < Me^{\alpha t} \quad \text{for all } t > t_0$$

(i.e., any continuous function, no matter how fast-growing, is bounded above by an exponential function for sufficiently large values of t.)

Hence, choose M, α such when $t_0 = 0$: $\left|f(t)\right| < Me^{\alpha t}$ for all $t > 0$

$$\left|L[f(t)]\right| = \left|\int_0^\infty f(t)e^{-st}\,dt\right| \le \int_0^\infty \left|f(t)\right|e^{-st}\,dt < \int_0^\infty Me^{\alpha t}e^{-st}\,dt = M\int_0^\infty e^{-(s-\alpha)t}\,dt$$

So: $$\lim_{d\to\infty} M\int_0^d e^{-(s-\alpha)t}\,dt = M\lim_{d\to\infty}\left\{\left.\tfrac{e^{-(s-\alpha)t}}{(\alpha-s)}\right|_0^d\right\} = M\left\{\lim_{d\to\infty}\tfrac{e^{-(s-\alpha)d}}{(\alpha-s)} - \tfrac{e^0}{(\alpha-s)}\right\}$$
$$= M\left\{0 - \tfrac{1}{(\alpha-s)}\right\} = \tfrac{M}{(s-\alpha)}$$

Hence: $$\lim_{s\to\infty} F(s) \le \lim_{s\to\infty}\left|F(s)\right| < \lim_{s\to\infty}\tfrac{M}{(s-\alpha)} = 0$$
$$\therefore \lim_{s\to\infty} F(s) = 0$$

The above Lemma gives us a valuable insight into the unique aspects of s – space (the domain space where the LTs live, i.e. the domain of the functions $F(s)$, given that t – space is the domain space where $f(t)$ live, i.e. the arguments of the LTs). Its 'asymptotic' behavior is such that it always vanishes, i.e. F is non-zero on a *finite* interval.

We can use Lemma 3b.1 to prove, for example, Thm 10:

$$\lim_{t\to 0^+} f(t) = \lim_{s\to\infty} sF(s), (where: F(s) = L[f(t)])$$

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[1] This Lemma can be generalized also to piecewise continuous functions.

Note1: we're taking a right-hand limit on the left hand side (in t-space) since the domain of f is nonnegative.

Proof: According to Thm2, for any differentiable (therefore continous) f:

$$L\left[\tfrac{d}{dt} f(t)\right] = sL\left[f(t)\right] - f(0)$$

Now, obviously, (since f is continuous[2]): $f(0) = \lim_{t\to 0^+} f(t)$

Hence: $$L\left[\tfrac{d}{dt} f(t)\right] = sL\left[f(t)\right] - f(0) = sL\left[f(t)\right] - \lim_{t\to 0^+} f(t)$$

Taking the $s \to \infty$ limit of both sides:

$$\lim_{s\to\infty} L\left[\tfrac{d}{dt} f(t)\right] = \lim_{s\to\infty} sL\left[f(t)\right] - \lim_{t\to 0^+} f(t)$$

Note 2: The last term on the right hand side is a constant with respect to s. Therefore its limiting value is the same throughout.[3]

But according to Lemma3b.1, the left hand side vanishes, if we assume the derivative of f is continuous.

Hence: $$\begin{aligned} &\lim_{s\to\infty} L\left[\tfrac{d}{dt} f(t)\right] = 0 = \lim_{s\to\infty} sL\left[f(t)\right] - \lim_{t\to 0^+} f(t) \\ &\therefore \lim_{t\to 0^+} f(t) = \lim_{s\to\infty} sL\left[f(t)\right] \end{aligned}$$

<u>Thm 11</u> is the analogous version, this time when: $s \to \infty$, and $t \to 0^+$

Both Theorems 10 & 11 should be viewed as tracking the limiting behavior of f in t-space and in reciprocal s-space simultaneously. Their seeming redundancy has a theoretical and a practical value. In theory, what makes their results interesting is giving the insight into the corresponding adjustment one needs to make in s –space (one must multiply the LT of f by s), so it's not just a simple matter of a strict correspondence between t and $^1/_s$. Their results also have practical value since if we know the LT of f, but if it's difficult to obtain the inverse LT (i.e., f), we can bypass having to if all we're interested in obtaining are the initial ($t \to 0$) and asymptotic ($t \to \infty$) conditions of f.

<u>Thm 9 (Periodic function thm)</u>:

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[2] Recall from CalcI: for any continuous function: $\lim_{x\to a} f(t) = f(a)$ for any a: $\infty < a < \infty$. (Note that $\lim_{x\to a} f(t)$ if and only if $\lim_{x\to a+} f(t) = \lim_{x\to a-} f(t)$. If we define the domain of f to be $[c, \infty)$, however, then the condition for f 's continuity at the endpoint c becomes: $\lim_{x\to c+} f(t) = f(c)$.

[3] I.e., for any constant-valued function $f(t) = k$: $\lim_{x\to a} f(t) = \lim_{x\to\infty} f(t) = k$, for all a: $\infty < a < \infty$.

If *f* is periodic, i.e. *f*(*t* + *p*) = *f*(*t*), then: $L[f(t)] = \frac{\int_0^p f(t)e^{-st}\,dt}{\left(1-e^{-ps}\right)}$

Proof:

$$L[f(t)] = \int_0^\infty f(t)e^{-st}\,dt = \int_0^p f(t)e^{-st}\,dt + \int_p^{2p} f(t)e^{-st}\,dt + \int_{2p}^{3p} f(t)e^{-st}\,dt + \ldots + \int_{np}^{(n+1)p} f(t)e^{-st}\,dt + \ldots$$

In the second integral, perform the *u* – substitution: *u* = *t* - *p*, then: *du* = *dt* and $t = u + p$. So: $\int_p^{2p} f(t)e^{-st}\,dt = \int_{u(p)}^{u(2p)} f(u+p)e^{-s(u+p)}\,du = e^{-sp}\int_0^p f(u)e^{-su}\,du$ (since *f*(*u* + *p*) = *f*(*u*))

In the third integral, perform the *u* – substitution: *u* = *t* - 2*p*, then: *du* = *dt* and $t = u + 2p$. So: $\int_{2p}^{3p} f(t)e^{-st}\,dt = \int_{u(2p)}^{u(3p)} f(u+2p)e^{-s(u+2p)}\,du = e^{-2sp}\int_0^p f(u)e^{-su}\,du$
(since *f*(*u* +2*p*) = *f*(*u*))

etc…

In the *n*-th integral, perform the *u* – substitution: *u* = *t* - *np*, then: *du* = *dt* and $t = u + np$. So: $\int_{np}^{(n+1)p} f(t)e^{-st}\,dt = \int_{u(np)}^{u((n+1)p)} f(u+np)e^{-s(u+np)}\,du = e^{-nsp}\int_0^p f(u)e^{-su}\,du$
(since *f*(*u* +*np*) = *f*(*u*))

Hence:

$$L[f(t)] = \int_0^\infty f(t)e^{-st}\,dt = \int_0^p f(t)e^{-st}\,dt + \int_p^{2p} f(t)e^{-st}\,dt + \int_{2p}^{3p} f(t)e^{-st}\,dt + \ldots + \int_{np}^{(n+1)p} f(t)e^{-st}\,dt + \ldots$$

$$= \int_0^p f(t)e^{-st}\,dt + e^{-sp}\int_0^p f(u)e^{-su}\,du + e^{-2sp}\int_0^p f(u)e^{-su}\,du + \ldots + e^{-nsp}\int_0^p f(u)e^{-su}\,du + \ldots$$

Now, because *u* is just a dummy-index of integration, we can re-name it as *t*. Then:

$$L[f(t)] = \int_0^p f(t)e^{-st}\,dt + e^{-sp}\int_0^p f(t)e^{-st}\,dt + e^{-2sp}\int_0^p f(t)e^{-st}\,dt + \ldots + e^{-nsp}\int_0^p f(t)e^{-st}\,dt + \ldots$$

$$= \left\{1 + e^{-sp} + e^{-2sp} + \ldots + e^{-nsp} + \ldots\right\}\int_0^p f(t)e^{-st}\,dt = \frac{1}{1-e^{-sp}}\int_0^p f(t)e^{-st}\,dt$$

(Since the series $1+e^{-sp}+e^{-2sp}+...$ is an infinite geometric series, with $r = e^{-sp}$. Moreover, since $r = e^{-sp} < 1$, then the series converges to: $\frac{1}{1-r} = \frac{1}{1-e^{-sp}}$)

Thm 12 (Convolution Thm).

For :

$$F(s) = L[f(t)], G = L[g(t)]$$

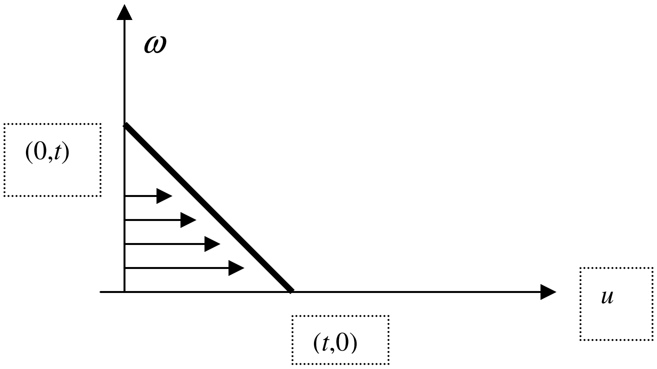
$$\therefore F(s)G(s) = L[(f * g)], where: (f * g) \equiv \int_0^t f(u)g(t-u)du = \int_0^t f(t-u)g(u)du$$

Proof:

$$F(s)G(s) = \int_0^\infty f(\omega)e^{-s\omega}d\omega \int_0^\infty g(u)e^{-su}du = \int_0^\infty \int_0^\infty f(\omega)g(u)e^{-s(\omega+u)}d\omega du$$

Let $t = \omega + u$. Then: $\omega = t - u,\ d\omega = dt$.

Furthermore, the limits of integration with respect to u are: $u = 0$ to $u = t$. To understand why, consider the figure below:



To see this (informally) observe that in the (ω, u) coordinate system, the line: $\omega = t - u$ has u-intercept (t,0) and ω-intercept: (0,t). Hence, as the inscribed arrows in the triangle suggest, we can fill up the entire first quadrant by traveling first from $u = 0$ to $u = t$, and then stacking these arrows vertically from $\omega = 0$ to $\omega = t$. If we let $\omega \rightarrow \infty$, then the entire space is filled. (To understand this more formally requires the tools of CalcIII, i.e. conformal mappings and the Jacobian.)

So:

$$F(s)G(s)=\int_0^\infty\int_0^\infty f(\omega)g(u)e^{-s(\omega+u)}du\,d\omega=\int_0^\infty\int_0^t f(t-u)g(u)e^{-st}du\,dt=\int_0^\infty e^{-st}\left\{\int_0^t f(t-u)g(u)du\right\}dt$$

$$=L\left[\int_0^t f(t-u)g(u)du\right]=L[(f*g)]$$

Note: If g were initially expressed in terms of dummy variable ω and f in terms of u, we'd arrive at the expression:

$$F(s)G(s)=L\left[\int_0^t g(t-u)f(u)du\right]=L[(g*f)]\Rightarrow(f*g)=\int_0^t f(t-u)g(u)du=\int_0^t g(t-u)f(u)du=(g*f)$$

i.e., the Convolution with respect to two functions is a commutative operation

Selected Examples

- Example (Problem 3, p 41 Sheng)

For $|\sin t|$, $p = \pi$.

So according to Thm 9: $L[|\sin t|]=\dfrac{1}{1-e^{-\pi s}}\displaystyle\int_0^\pi e^{-st}\sin t\,dt$

To evaluate the integral, you can use Formula 30 (Appendix A7—this was also derived using integration by parts in Handout 1a.) However, you could also use Euler's Theorem:

$$L[|\sin t|]=\frac{1}{1-e^{-\pi s}}\int_0^\pi e^{-st}\left(\frac{e^{it}-e^{-it}}{2i}\right)dt=\frac{1}{2i\left(1-e^{-\pi s}\right)}\left\{\int_0^\pi e^{(i-s)t}dt-\int_0^\pi e^{-(i+s)t}dt\right\}$$

$$=\frac{i}{2\left(1-e^{-\pi s}\right)}\left\{\int_0^\pi e^{-(i+s)t}dt-\int_0^\pi e^{(i-s)t}dt\right\}$$

(since: $^{1}/_{i} = -i$)

$$\frac{i}{2\left(1-e^{-\pi s}\right)}\left\{\int_0^\pi e^{-(i+s)t}dt-\int_0^\pi e^{(i-s)t}dt\right\}=\frac{i}{2\left(1-e^{-\pi s}\right)}\left\{-\left.\frac{e^{-(i+s)t}}{(i+s)}\right|_0^\pi-\left.\frac{e^{(i-s)t}}{(i-s)}\right|_0^\pi\right\}$$

$$=\frac{i}{2\left(1-e^{-\pi s}\right)}\left\{\frac{1-e^{-i\pi}e^{-s\pi}}{(i+s)}+\frac{-e^{i\pi}e^{-s\pi}+1}{(i-s)}\right\}$$

But according to Euler's Thm: $e^{\pm i\pi}=\cos\pi\pm i\sin\pi=-1$

So:

$$\frac{i}{2\left(1-e^{-\pi s}\right)}\left\{\frac{1-e^{-i\pi}e^{-s\pi}}{(i+s)}+\frac{1-e^{i\pi}e^{-s\pi}}{(i-s)}\right\}=\frac{i}{2\left(1-e^{-\pi s}\right)}\left\{\frac{1+e^{-s\pi}}{(i+s)}+\frac{e^{-s\pi}+1}{(i-s)}\right\}$$

$$=\frac{i}{2\left(1-e^{-\pi s}\right)}\left\{\left[\frac{1}{i+s}+\frac{1}{i-s}\right]+e^{-s\pi}\left[\frac{1}{i+s}+\frac{1}{i-s}\right]\right\}=\frac{i}{2\left(1-e^{-\pi s}\right)}\left\{\left(1+e^{-\pi s}\right)\left(\tfrac{1}{i+s}+\tfrac{1}{i-s}\right)\right\}$$

$$=\frac{i\left(1+e^{-\pi s}\right)}{2\left(1-e^{-\pi s}\right)}\left[\frac{i-s+i+s}{\left(-1-s^2\right)}\right]=\frac{i\left(1+e^{-\pi s}\right)}{2\left(1-e^{-\pi s}\right)}\left[\frac{-2i}{\left(s^2+1\right)}\right]=\frac{-i^2\left(1+e^{-\pi s}\right)}{\left(1-e^{-\pi s}\right)\left(s^2+1\right)}=\frac{\left(1+e^{-\pi s}\right)}{\left(1-e^{-\pi s}\right)\left(s^2+1\right)}$$

- <u>Example (5c, p 41 Sheng)</u>

Use **Thm 10** to verify in the case of $\lim_{t\to 0+} f(t)$, where: $f(t) = 2t - 3\sin 2t$

In *t*-space: $\lim_{t\to 0}\{2t-3\sin 2t\}=2\lim_{t\to 0} t-3\lim_{t\to 0}\sin 2t=0-0=0$

In *s*-space:

$$\lim_{s\to\infty} sF(s), F(s)=L[2t-3\sin 2t]=2L[t]-3L[\sin 2t]=2\cdot\tfrac{1!}{s^2}-3\cdot\tfrac{2}{s^2-4}$$
$$=\tfrac{2}{s^2}-\tfrac{6}{s^2-4}\Rightarrow sF(s)=\tfrac{2}{s}-\tfrac{6s}{s^2-4}=\tfrac{2s^2-8-6s^2}{s^3-4s}=\tfrac{-2\left(3s^2+4\right)}{s^3-4s}$$
$$\therefore \lim_{s\to\infty} sF(s)=\lim_{s\to\infty}\tfrac{-2\left(3s^2+4\right)}{s^3-4s}=-2\lim_{s\to\infty}\frac{\frac{s^2}{s^3}+\frac{4}{s^3}}{1-\frac{4s}{s^3}}=-2\lim_{s\to\infty}\frac{\frac{1}{s}+\frac{4}{s^3}}{1-\frac{4}{s^2}}=-2\cdot\left(\frac{0}{1}\right)=0$$

Therefore we've verified the result of **Thm10**

- <u>Example (6d, p 42 Sheng)</u>

Use **Thm 11** to verify in the case of $\lim_{t\to\infty} f(t)$, where: $f(t) = 4e^{-3t} - 10e^{-t}\sin 2t$

In *t*-space:
$\lim_{t\to\infty}\left\{4e^{-3t}-10e^{-t}\sin 2t\right\}=4\lim_{t\to\infty} e^{-3t}-10\lim_{t\to\infty}(e^{-t}\sin 2t)=4\cdot 0-10\cdot 0=0$

In *s*-space:

$$\lim_{s\to\infty} sF(s), F(s)=L\left[4e^{-3t}-10e^{-t}\sin 2t\right]=4L\left[e^{-3t}\right]-10L\left[e^{-t}\sin 2t\right]=4\cdot\tfrac{1}{s+3}-10G(s+1)$$
$$(G(s)=L[\sin 2t]=\tfrac{2}{s^2+4}\Rightarrow G(s+1)=\tfrac{2}{(s+1)^2+4}$$
$$\therefore F(s)=\tfrac{4}{s+3}-\tfrac{20}{(s+1)^2+4}\Rightarrow sF(s)=\tfrac{4s}{s+3}-\tfrac{20s}{(s+1)^2+4}$$
$$\therefore \lim_{s\to 0} sF(s)=\lim_{s\to 0}\left\{\tfrac{4s}{s+3}-\tfrac{20s}{(s+1)^2+4}\right\}=\tfrac{0}{3}-\tfrac{0}{5}=0$$

- <u>Example (7b, pg 42 Sheng)</u>

Use Thm 12 to find the inverse LT of $\dfrac{1}{\left(s^2+1\right)^2}$

$$F(s)G(s)=\frac{1}{\left(s^2+1\right)}\cdot\frac{1}{\left(s^2+1\right)}\Rightarrow f(t)=L^{-1}\left[F(s)\right]=\sin t=g(t)=L^{-1}\left[G(s)\right]$$

Hence according to Thm12: $L^{-1}\left[F(s)G(s)\right]=(f*g)=\int_0^t \sin(t-u)\sin u\,du$

<u>Method1</u>: $\sin(t-u)\sin u=\frac{1}{2}\left[\cos(t-2u)-\cos t\right]$ (using product-to-sum rule)

$$\therefore f*g=\tfrac{1}{2}\int_0^t\left[\cos(t-2u)\right]du-\tfrac{1}{2}\int_0^t\cos t\,du\Rightarrow U(u)=(t-2u)\therefore du=-\tfrac{1}{2}dU$$

$$=-\tfrac{1}{4}\int_{U(0)}^{U(t)}\cos U\,dU-\tfrac{1}{2}\cos t\int_0^t du=-\tfrac{1}{4}\sin U\Big|_t^{-t}-\tfrac{1}{2}\cos t\,u\Big|_0^t=-\tfrac{1}{4}\left(-\sin t-\sin t\right)-\tfrac{t}{2}\cos t$$

$$=\tfrac{1}{2}\sin t-\tfrac{t}{2}\cos t=\tfrac{1}{2}\left[\sin t-t\cos t\right]$$

<u>Method 2 (more cumbersome, but a review of some simple integration and other trig formulae)</u>

$$\sin(t-u)\sin u=\sin t\cos u\sin u-\cos t\sin^2 u$$

$$\int_0^t\sin(t-u)\cos u\,du=\sin t\int_0^t\sin u\cos u\,du-\cos t\int_0^2\sin^2 u\,du$$

$$=\sin t\cdot\tfrac{1}{2}\sin^2 u\Big|_0^t-\cos t\cdot\tfrac{1}{2}\int_0^t(1-\cos 2t)dt=\tfrac{1}{2}\left\{\sin^3 t-t\cos t+\cos t\int_0^t\cos 2t\,dt\right\}$$

$$=\tfrac{1}{2}\left\{\sin^3 t-t\cos t+\cos t\cdot\tfrac{1}{2}\sin 2u\Big|_0^2\right\}=\tfrac{1}{2}\left\{\sin^3 t-t\cos t+\cos t\cdot\tfrac{1}{2}\sin 2t\right\}$$

$$=\tfrac{1}{2}\left\{\sin^3 t-t\cos t+\cos t\cdot\tfrac{1}{2}\cdot 2\sin t\cos t\right\}=\tfrac{1}{2}\left\{\sin^3 t-t\cos t+\sin t\cos^2 t\right\}$$

$$=\tfrac{1}{2}\left\{\sin t\left(\sin^2 t+\cos^2 t\right)-t\cos t\right\}=\tfrac{1}{2}\left[\sin t-t\cos t\right]$$